SMOOTH FRÉCHET GLOBALIZATIONS OF HARISH-CHANDRA MODULES

JOSEPH BERNSTEIN AND BERNHARD KRÖTZ

1. **Introduction**

Let G be a linear reductive real Lie group G with Lie algebra $\mathfrak{g}$[1]. Let us fix a maximal compact subgroup K of G. The representation theory of G admits an algebraic underpinning encoded in the notion of a *Harish-Chandra module*.

By a Harish-Chandra module we shall understand a finitely generated $(\mathfrak{g}, K)$-module with finite K-multiplicities. Let us denote by $\mathcal{H}C$ the category whose objects are Harish-Chandra modules and whose morphisms are linear $(\mathfrak{g}, K)$-maps. By a *globalization* of a Harish-Chandra module V we understand a representation (π, E) of G such that the K-finite vectors of E are isomorphic to V as a $(\mathfrak{g}, K)$-module.

Let us denote by $\mathcal{SAF}$ the category whose objects are smooth admissible moderate growth Fréchet representations of G with continuous linear G-maps as morphisms. We consider the functor:

$$\mathcal{F} : \mathcal{SAF} \to \mathcal{H}C, \quad E \mapsto E_K := \{K - \text{finite vectors of E}\}.$$

The Casselman-Wallach theorem ([3] or [12], Sect. 11) asserts that $\mathcal{F}$ is an equivalence of categories. To phrase it differently, each Harish-Chandra module V admits a unique $\mathcal{SAF}$-globalization (π, V^∞). Moreover,

$$V^\infty = \pi(\mathcal{S}(G))V$$

where $\mathcal{S}(G)$ is the Schwartz-algebra of rapidly decreasing functions on G, and $\pi(\mathcal{S}(G))V$ stands for the vector space spanned by $\pi(f)v$ for $f \in \mathcal{S}(G)$, $v \in V$. In particular, V is irreducible if and only if V^∞ is an algebraically simple $\mathcal{S}(G)$-module.

One objective of this paper is to give an elementary proof of this fact. Our approach starts with a thorough investigation of the topological nature of $\mathcal{SAF}$-globalizations. A key result here is that every $\mathcal{SAF}$-globalization is nuclear. The established topological properties allow us to define *minimal* and *maximal* $\mathcal{SAF}$-globalizations of a Harish-Chandra module in a canonical way. We then proceed to show that the minimal and maximal globalization coincide. The language developed in this paper readily implies that minimal and maximal globalization coincide for representations of the discrete series. Next we consider spherical principal series representations of G with their canonical Hilbert-globalizations as subspaces of $L^2(K)$. For such representations we define a Dirac-type sequence and establish uniform lower bounds

---

[1]Throughout this text Lie groups will be denoted by upper case Latin letters, G, K, N ... , and their corresponding Lie algebras by lower case German letters $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{n}$ etc.

for K-finite matrix coefficients (see Theorem 7.3 below). A corollary of these lower bounds is that minimal and maximal globalizations for these type of representations coincide. The case of arbitrary Harish-Chandra modules will be reduced to this to these two basic cases.

We wish to emphasize that our lower bounds are locally uniform in representation parameters which allows us to prove a version of the Casselman-Wallach theorem with holomorphic dependance on representation parameters (see Section 7). This for instance is useful for the theory of Eisenstein series.

Acknowledgment: We wish to thank Birgit Speh and Henrik Schlichtkrull for several very useful discussions.

2. Basic representation theory

In this section we collect some basic notions of representation theory.

2.1. Representations on topological vector spaces

All topological vector spaces E considered in this paper are understood to be locally convex. We denote by $\mathrm{Gl}(E)$ the group of all topological linear isomorphisms of E.

Let G be a Lie group and E a topological vector space. By a *representation* (π, E) of G on E we understand a homomorphism $\pi : G \to \mathrm{Gl}(E)$ such that the resulting action $G \times E \to E$ is continuous. We emphasize that continuity is requested in both variables. For an element $v \in E$ we shall denote by

$$\gamma_v : G \to E, \quad g \mapsto \pi(g)v$$

the corresponding continuous orbit map.

Lemma 2.1. *Let G be a Lie group, E a topological vector space, $\pi : G \to \mathrm{Gl}(E)$ a group homomorphism and $G \times E \to E$ the resulting action. Then the following statements are equivalent:*

(i) *The action $G \times E \to E$ is continuous, i.e. (π, E) is a representation.*

(ii) (a) *There exists a dense subset $E_0 \subset E$ such that for all $v \in E_0$ the orbit map $\gamma_v : G \to E$ is continuous.*

(b) *For all compact subsets Q of G the set $\{\pi(g) \mid g \in Q\}$ is an equicontinuous set of linear endomorphisms of E.*

Proof. (i) $\Rightarrow$ (ii): (a) is clear and we move on to (b). We are going to exploit the continuity of the action $G \times E \to E$ in the points $(g, 0)$ for $g \in Q$. Let V be an open neighborhood of 0 in E. Then for all $g \in Q$

we find a neighborhood U_g of g and a neighborhood W_g of 0 in E such that $\pi(U_g)W_g \subset V$. As Q is compact, finitely many U_g cover Q, say $U_{g_1}, \ldots, U_{g_n}$. Then $W := \bigcap_{i=1}^n W_{g_i}$ is a neighborhood of 0 in E with $\pi(Q)W \subset V$. As V was arbitrary, equicontinuity follows.

(ii) $\Rightarrow$ (i): It is enough to establish continuity at the points $(\mathbf{1}, v_0)$ for $v_0 \in E$. Neighborhoods of v_0 in E have the form $v_0 + V$ where V is convex neighborhood of 0 in E. We have to show that there is an neighborhood U of 0 in E and a neighborhood B of $\mathbf{1}$ of G such that $\pi(B)(v_0 + U) \subset v_0 + V$.

Let B_1 be a compact neighborhood of $\mathbf{1}$ in G. By equicontinuity we find a symmetric neighborhood U of 0 in E such that $\pi(B_1)U \subset \frac{1}{4}V$. Choose $v_1 \in E_0$ such that $v_0 \in v_1 + U$. As the orbit map γ_{v_1} is continuous, we find a compact neighborhood B_2 of $\mathbf{1}$ in G such that $\pi(B_2)v_1 \subset v_1 + \frac{1}{4}V$. Set $B := B_1 \cap B_2$. Then

$$\pi(B)(v_0+U) \subset \pi(B)(v_1+2U) \subset v_1+\frac{1}{4}V+\frac{1}{2}V \subset v_0+U+\frac{3}{4}V \subset v_0+V\,.$$

$$\square$$

Remark 2.2. *Suppose that (π, E) is a representation on a semi-normed space E. Then all operator semi-norms of $\pi(g)$ are locally bounded in $g \in G$.*

If $\pi : G \to \mathrm{Gl}(E)$ is a group homomorphism, then we say π is *locally equicontinuous* if condition (b) in the Lemma above is satisfied.

If (π, E) is a representation, then we call a continuous semi-norm ρ on E a *G-continuous semi-norm*, if $G \times (E, \rho) \to (E, \rho)$ is continuous. Here (E, ρ) stands for the vector space E endowed with the topology induced from the semi-norm ρ.

Remark 2.3. *Let ρ be a G-continuous semi-norm on a representation module E. The kernel of ρ, say N, is a subspace and ρ induces a norm on the quotient E/N. As $G \times (E, \rho) \to (E, \rho)$ is continuous, it follows that G preserves N and induces a representation on the normed space E/N. In view Lemma 2.1 this representation naturally extends to a representation on the Banach completion of E/N.*

Let us call a G-continuous norm (resp. semi-norm) ρ on a representation module (π, E) a *G-continuous Hilbert norm (rep. semi-norm)* if ρ is defined by a positive definite (resp. semi-definite) Hermitian form $\langle \cdot, \cdot \rangle$ on E, i.e. $\rho(v)^2 = \langle v, v \rangle$ for all $v \in E$.

Let (π, E) be a representation of G. If E is a Banach (Hilbertian, Fréchet) space, then we speak of a *Banach (Hilbertian, Fréchet) representation* of G. We call (π, E) an *F-representation* if E is a Fréchet

space whose topology is induced by a countable family of G-continuous semi-norms $(\rho^n)_{n \in \mathbb{N}}$.

2.1.1. *Smooth vectors and smooth representations.* Let (π, E) be a representation of G. We call a vector $v \in E$ *smooth* if γ_v is a smooth map and denote by E^∞ the space of all smooth vectors. Note that $\mathcal{U}(\mathfrak{g})$, the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of G, acts naturally on E^∞.

If p is a continuous semi-norm on E and $u \in \mathcal{U}(\mathfrak{g})$, define the Sobolev semi-norm p_u on E^∞ by

$$p_u(v) := p(d\pi(u)v) \qquad (v \in E^\infty).$$

We topologize E^∞ with regard to all Sobolev-norms of continuous semi-norms on E. Note that this turns E^∞ into a locally convex topological vector space. As a result π induces a representation on E^∞ (usually also denoted by π).

Remark 2.4. *The vector space $\mathcal{U}(\mathfrak{g})$ admits a countable basis, say $u_1, u_2, \ldots$. The natural filtration by degree yields a surjective map $\mathcal{U}(\mathfrak{g}) \to \mathbb{N}_0$, $u \mapsto \deg u$. Given p as above and $k \in \mathbb{N}_0$ we set*

$$Sp_k(v) := \sum_{\deg u_j \leq k} p_{u_j}(v) \qquad (v \in E^\infty)$$

and note that the families $(p_u)_{u \in \mathcal{U}(\mathfrak{g})}$ and $(Sp_k)_{k \in \mathbb{N}_0}$ induce the same topology on the vector space E.

Let us provide specific Sobolev semi-norms which define the topology on E^∞. For that let us fix a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ and define

$$\Delta := X_1^2 + \ldots + X_n^2 \in \mathcal{U}(\mathfrak{g}).$$

If p is a continuous semi-norm on a representation module E, then we define its $2k$-th Sobolev norm by

$$p_{2k}(v) := p(d\pi((k \cdot \mathbf{1} - \Delta)^k)v) \qquad (v \in E^\infty).$$

In the context we record a recent result (see [7], Rem. 5.6 (b)) of which we shall give an independent proof at this place:

Lemma 2.5. *Let G be Lie group and (π, E) be an F-representation of G. Let $(\rho^n)_{n \in \mathbb{N}}$ be a defining family of G-continuous semi-norms for (π, E). Then the topology of E^∞ is defined by the family of Sobolev semi-norms $(\rho^n_{2k})_{n,k \in \mathbb{N}}$.*

Proof. For every representation (π, E) on a complete space E, the convolution algebra $\mathcal{M}(G)$ of compactly supported Borel-measures acts continuously on E via

$$\pi(\mu)v := \int_G \pi(g)v \, d\mu(g) \qquad (\mu \in \mathcal{M}(G), v \in E)\,.$$

Let now (π, E) be an F-representation of G. As usual we write $\delta_{\mathbf{1}}$ for the Dirac-delta distribution on G. We let G act on functions on G by the left regular representation and obtain in this way for every $u \in \mathcal{U}(\mathfrak{g})$ a right-invariant distribution $u * \delta_{\mathbf{1}}$. Basic elliptic PDE-theory yields for every $u \in \mathcal{U}(\mathfrak{g})$ a $k \in \mathbb{N}$ and continuous compactly suported functions f_1, f_2 on G such that

$$(k \cdot \mathbf{1} - \Delta)^k f_1 + f_2 = u * \delta_{\mathbf{1}}\,.$$

We apply this idendity to a smooth vector $v \in E^\infty$ and arrive at

$$d\pi(u)v = d\pi(k \cdot \mathbf{1} - \Delta)^k \pi(f_1)v + \pi(f_2)v\,.$$

In particular, if p is a G-continuous semi-norm on E we get

$$p_u(v) \leq p_k(\pi(f_1)v) + p(\pi(f_2)v)\,.$$

Note that $p(\pi(f_2)v) \leq Cp(v)$ for some $C > 0$ as p is G-continuous; likewise we obtain $p_k(\pi(f_1)v) \leq C \sum_{j \leq k} p_j(v)$ by elliptic regularity. $\square$

We call a representation (π, E) *smooth* if $E = E^\infty$ as topological vector spaces. In the sequel we will abbreviate and call smooth F-representations simply SF-representations.

Example 2.6. *Suppose that (π, E) is a Banach representation. Then E^∞ is a Fréchet space and (π, E^∞) is an SF-representation of G.*

2.1.2. *Dual representations.* If E is a topological vector space, then we denote by E^* its dual, i.e. the space of continuous linear forms on E. We endow E^* with the strong topology, i.e. the topology of bounded convergence.

If (π, E) is a representation, then we obtain a homomorphism

$$\pi^* : G \to \mathrm{Gl}(E^*), \quad \pi^*(g)(\lambda) := \lambda \circ \pi(g^{-1})\,.$$

From the local equicontinuity of π the local equicontinuity of π^* follows. However, orbit maps for π^* might fail to be continuous as the following example shows.

Example 2.7. *Let $G = \mathbb{S}^1$ and (π, E) be the left regular representation of G on $E = L^1(G)$. Then π^* identifies with the right regular action of G on $E^* = L^\infty(G)$ where not all orbit maps are continuous.*

It depends on the nature of the locally convex space E whether π^* defines a representation on E^*. For instance, if E is reflexive, then (π^*, E^*) is a representation (see [13]).

The fact that orbit maps for π^* might fail to be continuous can be dealt with in the following way. We define the *continuous dual* E_c^* of E as those vectors $\lambda \in E^*$ for which the orbit map $\gamma_\lambda : G \to E^*$ is continuous. Lemma 2.1 implies that E_c^* is a closed G-invariant subspace of E^*. We call (π^*, E_c^*) the continuous dual representation of (π, E). For instance in the example from above with $E = L^1(\mathbb{S}^1)$ we would have $E_c^* = C(\mathbb{S}^1) \subset E^* = L^\infty(\mathbb{S}^1)$.

If (π, E) is a Banach representation of G and p is a defining norm of E, then we define its dual norm p^* on E by

$$p^*(\lambda) := \sup_{p(v) \leq 1} |\lambda(v)| \qquad (\lambda \in E^*).$$

We write p_c^* for the restriction of p^* to E_c^*. Recall that E maps isometrically into its bidual E^{**}. Likewise the inclusion $E_c^* \to E^*$ yields a contractive projection $E^{**} \to (E_c^*)^*$. It follows that

$$(2.1) \qquad (p_c^*)^*(v) \leq p(v) \qquad (v \in E).$$

2.2. **Growth of representations**

Let us fix a left-invariant Riemannian metric $\mathbf{g}$ on G. Associated to $\mathbf{g}$ is the distance function $d(g, h)$, i.e. the minimum length of piecewise smooth curve joining g and h in G. Note that $d(\cdot, \cdot)$ is left G-invariant and hence can be recovered from

$$d(g) := d(g, \mathbf{1}) \qquad (g \in G).$$

Note that $d(g)$ is *subadditive*, i.e. $d(gh) \leq d(g) + d(h)$ for all $g, h \in G$.

By a weight on G we understand a function $w : G \to \mathbb{R}^+$ such that

- w is locally bounded.
- w is *submultiplicative*, i.e. $w(gh) \leq w(g)w(h)$ for all $g, h \in G$.

An important fact is that for any weight w there exists a constant $C > 0$ such that

$$(2.2) \qquad \log w(g) \leq C d(g) + C \qquad (g \in G),$$

see [6], Lemme 2.

Let (π, E) be a Banach representation. Then the uniform boundedness principle implies that

$$w_\pi : G \to \mathbb{R}^+, \quad g \mapsto \|\pi(g)\|$$

is a weight. We call w_π the weight associated to (π, E). It follows from (2.2) that there exist a constant $C > 0$ such that

$$(2.3) \qquad \|\pi(g)\| \leq Ce^{C \cdot d(g)} \qquad (g \in G).$$

Following Casselman [3] we call a Fréchet representation (π, E) of *moderate growth* if for any semi-norm p on E there exists a semi-norm q on E and an integer $N > 0$ such that

$$p(\pi(g)v) \leq e^{Nd(g)}q(v)$$

for all $g \in G$.

Lemma 2.8. *For a Fréchet representation (π, E) the following statements are equivalent:*

 (i) (π, E) *is of moderate growth.*
 (ii) (π, E) *is an F-representation.*

Proof. In view of Remark 2.3 and (2.2) any Fréchet representation is of moderate growth.

Conversely, assume that (π, E) is of moderate growth and let p, q and $N > 0$ be as in the definition above. Then

$$\widetilde{p}(v) := \sup_{g \in G} \frac{p(\pi(g)v)}{e^{Nd(g)}}$$

defines a semi-norm on E such that

 • $p \leq \widetilde{p} \leq q$.
 • $\widetilde{p}(\pi(g)v) \leq e^{Nd(g)}\widetilde{p}(v)$ for all $g \in G$.

The first bulleted item implies that the $\widetilde{p}$ define the topology on E and the second bulleted item yields that $G \times (E, \widetilde{p}) \to (E, \widetilde{p})$ is continuous. $\square$

Recall that the category of Fréchet spaces is closed under taking closed subspaces and quotients by closed subspaces. The same holds for the category of F and SF-representations.

Lemma 2.9. *Let (π, E) be a F-representation and $F \subset E$ a closed G-invariant subspace. Then the corresponding quotient representation on E/F is a Fréchet representation. Moreover, if E is smooth, then E/F is smooth.*

Proof. Recall that the Fréchet topology on the quotient is induced from the corresponding quotient semi-norms. Let now q be a G-continuous semi-norm on E and

$$\overline{q}(v + F) := \inf_{w \in F} q(v + w) \qquad (v \in E)$$

be the corresponding quotient semi-norm. We claim that $G\times(E/F,\overline{q})\to$ $(E/F,\overline{q})$ is continuous. For that it is sufficient to show that the induced representation, say $\overline{\pi}$, is locally equicontinuous and that all orbit maps on E/F are continuous (Lemma 2.1). If $C>0$ is such that $q(\pi(g)v)\leq Ce^{Cd(g)}q(v)$, then it follows that $\overline{q}(\overline{\pi}(g)(v+F))\leq Ce^{Cd(g)}\overline{q}(v+F)$. Hence $\overline{\pi}$ is locally equicontinuous. Moreover

$$\overline{q}(\overline{\pi}(g)(v+F)-(v+F))\leq q(\pi(g)v-v)$$

and hence all induced orbit maps are continuous. We conclude that E/F is a Fréchet module. Moreover if E is smooth, then so is E/F. $\square$

2.3. Integration of representations and algebra actions

We fix a left Haar measure dg on G and recall (see [6], Lemme 1) that there exists a constant $c>0$ such that

$$\int_G e^{-cd(g)}\ dg<\infty\,.$$

Let us denote by $L^1(G)$ the Banach space of integrable functions on G. Note that $L^1(G)$ is a Banach algebra with multiplication given by convolution:

$$f*h(x)=\int_G f(g)h(g^{-1}x)\ dg\qquad(x\in G)$$

for $f,h\in L^1(G)$. We write $C_c^\infty(G)<L^1(G)$ for the subalgebra of test function on G.

If (π,E) is representation of G on a complete topological vector space, then we denote by by Π the corresponding algebra representation of $C_c^\infty(G)$:

$$(2.4)\qquad \Pi(f)v=\int_G f(g)\pi(g)v\ dg\qquad(f\in C_c^\infty(G),v\in E)\,.$$

Note that the defining vector valued integral actually converges as E is complete.

Depending on the type of the representation (π,E) larger algebras as $C_c^\infty(G)$ might act on E. For instance if (π,E) is a bounded Banach representation, then Π extends to a representation of $L^1(G)$. The natural algebra acting on a F-representation is the algebra of rapidly decreasing functions $\mathcal{R}(G)$ and the natural algebra acting on an SF-representation is the Schwartz algebra $\mathcal{S}(G)$.

The space of *rapidly decreasing functions* on G is defined as

$$\mathcal{R}(G) := \{f \in C(G) \mid (\forall n \in \mathbb{N}) \sup_{g \in G} e^{n \cdot d(g)} |f(g)| < \infty\}\,.$$

Let us emphasize that $\mathcal{R}(G)$ is independent of the particular choice of the left-invariant Riemannian metric on G. We write $L \otimes R$ for the regular representation of $G \times G$ functions on G:

$$(L \times R)(g_1, g_2) f(g) := f(g_1^{-1} g g_2)$$

for $g, g_1, g_2 \in G$ and $f \in C(G)$. The following properties of $\mathcal{R}(G)$ are easy to verify:

- $(L \otimes R, \mathcal{R}(G))$ is an F-representation of $G \times G$.
- $\mathcal{R}(G)$ is a Fréchet algebra under convolution.
- Any F-representation (π, E) of G integrates to a continuous algebra representation

(2.5) $$\mathcal{R}(G) \times E \to E, \quad (f, v) \mapsto \Pi(f)v,$$

  i.e. the E-valued integrals in (2.4) converge absolutely, the bilinear map (2.5) is continuous and $\Pi(f * h) = \Pi(f)\Pi(h)$ holds for all $f, h \in \mathcal{R}(G)$.

For $u \in \mathcal{U}(\mathfrak{g})$ we will abbreviate $L_u := dL(u)$ and likewise R_u for the derived representations. The smooth vectors of $(L \otimes R, \mathcal{R}(G))$ constitute the *Schwartz space*

$$\mathcal{S}(G) := \{f \in C^\infty(G) \mid \forall u, v \in \mathcal{U}(\mathfrak{g}), \forall n \in \mathbb{N}$$
$$\sup_{g \in G} e^{nd(g)} |L_u R_v f(g)| < \infty\}\,.$$

It is clear that $\mathcal{S}(G)$ is a Fréchet subalgebra of $\mathcal{R}(G)$ (see [11], Sect. 7.1 for a discussion in a wider context).

Remark 2.10. *For a function $f \in \mathcal{R}(G)$ the following assertions are equivalent: (1) f is $\mathcal{S}(G)$, i.e. f is $L \times R$-smooth; (2) f is R-smooth; (3) f is L-smooth. In fact, a left derivative L_u at a point $g \in G$ is the same as a right derivative $R_{\mathrm{Ad}(g)^{-1} u}$ at g. Now observe that $\|\mathrm{Ad}(g)\| \le C \cdot e^{Cd(g)}$ for all $g \in G$ and a fixed $C > 0$.*

As $\mathcal{R}(G)$ acts on all F-representations it follows that $\mathcal{S}(G)$ acts on all SF-representations.

Remark 2.11. *If (π, E) is a smooth Fréchet-representation, then $\Pi(C_c^\infty(G))E = E$ by Dixmier-Malliavin [5]. Assume in addition that (π, E) is an SF-representation. As $\mathcal{R}(G)$ acts on E and $\mathcal{R}(G) \supset \mathcal{S}(G) \supset C_c^\infty(G)$, we deduce that $\Pi(\mathcal{S}(G))E = \Pi(\mathcal{R}(G))E = E$.*

If $\mathcal{A}$ is an algebra and M is an $\mathcal{A}$-module, then we call M *non-degenerate* if $\mathcal{A}M = M$.

Proposition 2.12. *Let G be a Lie group. Then the following categories are equivalent:*

(i) *The category of SF-representations of G.*
(ii) *The category of non-degenerate continuous algebra representations of $\mathcal{S}(G)$ on Fréchet spaces.*

Proof. We already saw that every SF-representations (π, E) gives rise to a non-degenerate continuous algebra representation (Π, E) of $\mathcal{S}(G)$. Conversely let (Π, E) by a continuous non-degenerate algebra representation of $\mathcal{S}(G)$. Let us denote by $S(G)\hat{\otimes}_\pi E$ the projective tensor product of $\mathcal{S}(G)$ and E. Clearly $S(G)\hat{\otimes}_\pi E$ is a Fréchet space and we define an SF-module structure for G by

$$g \cdot (f \otimes v) := L(g)f \otimes v \qquad (g \in G, f \in \mathcal{S}(G), v \in E).$$

As Π is non-degenerate, E becomes a quotient of $S(G)\hat{\otimes}_\pi E$ and Lemma 2.9 completes the proof. $\qquad\qquad\square$

3. **Harish-Chandra modules**

From now on we assume that G is a linear reductive group. Let us fix a maximal compact subgroup K of G.

We call a K-module E *weakly admissible* if for all irreducible representations (τ, W) of K the multiplicity space $\mathrm{Hom}_K(W, E)$ is finite dimensional.

We call a representation (π, E) of G *weakly admissible* if E is admissible for the K-module structure induced by π.

By a $(\mathfrak{g}, K)$-module V we understand a module for $\mathfrak{g}$ and K such that:

- The K-action is algebraic, i.e. V is a union of finite dimensional algebraic K-modules.
- The derived action of K coincides with the action of $\mathfrak{g}$ restricted to $\mathfrak{k} := \mathrm{Lie} K$.
- The actions are compatible, i.e.

$$k \cdot X \cdot v = \mathrm{Ad}(k)X \cdot k \cdot v$$

for all $k \in K$, $X \in \mathfrak{g}$ and $v \in V$.

Note that if (π, E) is a weakly admissible Banach representation of G, then the space of K-finite vectors of E, say E_K, consists of smooth

vectors and is stable under $\mathfrak{g}$ – in other words E_K is a weakly admissible $(\mathfrak{g}, K)$-module.

Let us emphasize that a weakly admissible $(\mathfrak{g}, K)$-module is not necessary finitely generated as a $\mathfrak{g}$-module. For example the tensor product of two infinite dimensional highest weight modules for $\mathfrak{sl}(2, \mathbb{R})$ is admissible but not finitely generated as a $\mathfrak{g}$-module. This brings us to the notion of a *Harish-Chandra module* or *admissible $(\mathfrak{g}, K)$-module* by which we understand a $(\mathfrak{g}, K)$-module V such that one of the following equivalent conditions hold:

(i) V is weakly admissible and finitely generated as a $\mathfrak{g}$-module.
(ii) V is weakly admissible and $\mathcal{Z}(\mathfrak{g})$-finite. Here $\mathcal{Z}(\mathfrak{g})$ denotes the center of $\mathcal{U}(\mathfrak{g})$.
(iii) V is finitely generated as an $\mathfrak{n}$-module, where $\mathfrak{n}$ is a maximal unipotent subalgebra of $\mathfrak{g}$.

We will call a Fréchet representation (π, E) admissible if the underlying $(\mathfrak{g}, K)$-module E_K is admissible.

Given a Harish-Chandra module V we say that a representation (π, E) of G is a *globalization* of V, if the K-finite vectors E_K of E are smooth and isomorphic to V as a $(\mathfrak{g}, K)$-module.

Remark 3.1. *We caution the reader that there exist irreducible Banach representation (π, E) of G which are not admissible [9]. However, if $(\pi, \mathcal{H})$ happens to be unitary irreducible representation, then Harish-Chandra has shown that π is admissible.*

Remark 3.2. *Let V be a Harish-Chandra module and (π, E) a Banach globalization. Then*

$$\Pi(\mathcal{R}(G))V = \Pi(\mathcal{S}(G))V .$$

In order to see that we use a result of Harish-Chandra which asserts that for each $v \in V$ there exists a $K \times K$-finite $h \in C_c^\infty(G)$ such that $\Pi(h)v = v$. As $\mathcal{R}(G) * C_c^\infty(G) \subset \mathcal{S}(G)$ the asserted equality is established.

3.1. **Existence of globalizations**

Every Harish-Chandra module V admits a Hilbert globalization $\mathcal{H}$ (and hence an SF-globalization by taking the smooth vectors in $\mathcal{H}$).

To construct such a globalization let us fix an Iwasawa decomposition $G = NAK$ and write $P_{\min} = MAN$ for the associated minimal parabolic subgroup, where $M := Z_K(A)$ is the centralizer of A in K.

According to Casselman (see [11], Cor. 4.2.4) one can embed V into a principal series module $I := C^\infty(W \times_{P_{\min}} G)_K$ where W is a

finite dimensional module for $P_{\min}$. As a K-module I is isomorphic to $\mathbb{C}[W \times_M K]$ and one shows that $L^2(W \times_M K)$ defines a Hilbert globalization of I. The desired Hilbert globalization $\mathcal{H}$ of V is then obtained by taking the closure of V in $L^2(W \times_M K)$.

3.2. **Basic topological properties of globalizations**

In this subsection we prove a variety of basic topological results about globalizations of of Harish-Chandra modules.

3.2.1. *The continuous dual of a Banach-globalization.* In the introductory section we saw that the dual of a Banach representation (π, E) might not be continuous which brought us to the notion of a continuous dual. The continuous dual E_c^* was the largest closed subspace of E^* on which the dual action is continuous.

The case where (π, E) is a Banach globalization of a Harish-Chandra module V is of particular interest to us. Here the situaion is more well behaved: E_c^* is a globalization of the dual Harish-Chandra module and if E is a Hilbert space, then $E_c^* = E^*$.

Let us briefly recall the notion of "dual" in the category of Harish-Chandra modules. If V is a Harish-Chandra module, then we denote by V^* its algebraic dual and by $\widetilde{V} \subset V^*$ the K-finite vectors in V^*. Note that $\widetilde{V}$ is a $\mathfrak{g}$-submodule of V^*. As V is weakly admissible we readily obtain that

$$\widetilde{\widetilde{V}} = V \, .$$

As Harish-Chandra modules have finite length (cf. [11], Th. 4.2.6), it follows that $\widetilde{V}$ is again a Harish-Chandra module (cf. [11], Lemma 4.3.2). We refer to $\widetilde{V}$ as the *Harish-Chandra module dual to V*.

Lemma 3.3. *Let V be a Harish-Chandra module and $\widetilde{V}$ its dual. If (π, E) is a Banach globalization of V, then $\widetilde{V} \subset E_c^*$. In particular, (π^*, E_c^*) is a Banach globalization of $\widetilde{V}$.*

Proof. Let (π, E) be a Banach globalization of V. For a K-type $\tau \in \hat{K}$ let us consider the projection

$$\mathrm{pr}_\tau : E \to E[\tau] = V[\tau], \quad v \mapsto \frac{1}{d(\tau)} \int_K \overline{\chi_\tau(k)} \pi(k) v \; dk$$

on the τ-isotypical component. Here χ_τ refers to the character of τ. As pr_τ is continuous, the first assertion $\widetilde{V} \subset E_c^*$ follows. Finally a K-type of E_c^* does not vanish on V as V is dense in E. Thus the

K-finite vectors of E_c^* are contained in $\widetilde{V}$ and the proof of the lemma is complete. $\qquad\square$

Lemma 3.4. *Let $(\pi, \mathcal{H})$ be a Hilbert globalization of V, then $\mathcal{H}^* = \mathcal{H}_c^*$.*

Proof. From the previous lemma it follows that $\widetilde{V}$ is contained in $\mathcal{H}_c^*$. As V contains an orthonormal basis of $\mathcal{H}$, we deduce that $\widetilde{V}$ contains an orthonormal basis of $\mathcal{H}^*$. Hence $\widetilde{V}$ is dense in $\mathcal{H}^*$. The lemma follows. $\qquad\square$

3.2.2. *K-Sobolev norms.* We recall the notion of Sobolev semi-norms p_{2k} from the introductory section. For that we fixed a basis $X_1, \ldots, X_n$ to define the Laplacian element $\Delta = \sum_{j=1}^n X_j^2$. The choice of the basis is in fact irrevelant and henceforth we will use a specific basis which is suitable for us. Such a basis is constructed as follows. Let $\mathfrak{k}$ denote the Lie algebra of K and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the associated Cartan decomposition of $\mathfrak{g}$. We fix a non-degenerate invariant bilinear form $B(X, Y)$ on $\mathfrak{g}$ such that B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. If $\theta : \mathfrak{g} \to \mathfrak{g}$ is the Cartan-involution associated to $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, then $\langle X, Y \rangle = B(\theta(X), Y)$ is an inner product. Our choice of $X_1, \ldots, X_n$ will be such that $X_1, \ldots X_m$ forms an orthonormal basis of $\mathfrak{k}$ and $X_{m+1}, \ldots, X_n$ is an orthonormal basis of $\mathfrak{p}$.

With regard to our choice of basis we obtain a central element (Casimir element) in $\mathcal{Z}(\mathfrak{g})$ by setting

$$\mathcal{C} = -\sum_{j=1}^m X_j^2 + \sum_{j=m+1}^n X_j^2$$

and with $\Delta_\mathfrak{k} := \sum_{j=1}^m X_j^2$ we arrive at the familiar relation

$$\mathcal{C} = \Delta - 2\Delta_\mathfrak{k}.$$

Let p be a continuous semi-norm on an SF-module E of G. Then for $s \in \mathbb{R}$ we define its s-th K-Sobolev semi-norm by

$$p_{s,K}(v) := p((1 - \Delta_\mathfrak{k})^{\frac{s}{2}} v) \qquad (v \in E).$$

We note that this in fact defined as the action of $(\mathbf{1} - \Delta_\mathfrak{k})^{\frac{s}{2}}$ on E is realized by left convolution with a distribution $\Theta_s \in C^{-\infty}(K)$ on K and the observation that $C^{-\infty}(K)$ acts continuously on every SF-module.

Lemma 3.5. *Let V be a Harish-Chandra module with infinitesimal charcater and (π, E) be a Banach globalization of V with norm p. Then the topology on the SF-module E^∞ is induced by the K-Sobolev norms semi-norms $(p_{n,K})_{n \in \mathbb{N}}$.*

Proof. Combine Lemma 2.5 with $\Delta = \mathcal{C} + 2\Delta_{\mathfrak{k}}$ and note that $\mathcal{C}$ acts as a scalar on E^{∞}. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.2.3. *Nuclear structures on smooth vectors.* The goal of this section is to prove that the smooth vectors of a Banach-globalization carry the structure of a nuclear Fréchet space. To begin with let us provide some notations with regard to the maximal compact subgroup K.

Let us denote by $\hat{K}$ the set of equivalence classes of irreducible unitary representations of K. We often identify an equivalence class $[\tau] \in \hat{K}$ with a representative τ. If V is a K-module, then we denote by $V[\tau]$ its τ-isotypical part. Similarly we denote for $v \in V$ by v_{τ} its K-isotypical component.

If $\mathfrak{t} \subseteq \mathfrak{k}$ is a maximal torus, then we often identify τ with its highest weight in $i\mathfrak{t}^*$ (with respect to a fixed positive system). In particular, $|\tau| \geq 0$ will refer to the norm of τ with repect to the positive definite form $-B|_{\mathfrak{k}}$.

Proposition 3.6. *Let (π, E) be Banach globalization of a Harish-Chandra module V. Then E^{∞} is a nuclear Fréchet space.*

Proof. Let p be a defining G-continuous norm of E. Then the topology on the SF-module E^{∞} is induced by all Sobolev norms $(p_k)_{k \in \mathbb{N}_0}$. Let $q := p_n$ be a fixed Sobolev norm.

Note that there is an integer $N > 0$ such that

$$m(\tau) := \dim V[\tau] \leq (1 + |\tau|)^N$$

holds for all $\tau \in \hat{K}$. According to the Lemma below we find for each $\tau \in \hat{K}$ a basis $v_1^{\tau}, \ldots, v_{m(\tau)}^{\tau}$ of $V[\tau]$ and a basis $\lambda_1^{\tau}, \ldots, \lambda_{m(\tau)}^{\tau}$ of $V[\tau]^*$ such that

$$\mathrm{id}_{V[\tau]} = \sum_{j=1}^{m(\tau)} v_j^{\tau} \otimes \lambda_j^{\tau}$$

with $q(v_i^{\tau}) \leq m(\tau)$ and $q^*(\lambda_j^{\tau}) = 1$ for all i, j. Let now $k \in \mathbb{N}$ and $q_{k,K}(v) = q((\mathbf{1} - \Delta_{\mathfrak{k}})^{k/2}v)$ be the k-th K-Sobolev norm of q. Thus for all $v \in V[\tau]$ we have

$$q_{k,K}(v) = (|\tau + \rho_{\mathfrak{k}}|^2 - |\rho_{\mathfrak{k}}|^2 + 1)^{k/2} q(v) \,.$$

Thus for $k > N^2 + \mathrm{rank}\, K + 1$ we get that

$$\sum_{\tau \in \hat{K}} \sum_{j=1}^{m(\tau)} q(v_j^{\tau}) \cdot q_{k,K}^*(\lambda_j^{\tau}) < \infty$$

But this means that

$$\mathrm{id}_V = \sum_{\tau \in \hat{K}} \sum_{j=1}^{m(\tau)} v_j^\tau \otimes \lambda_j^\tau$$

is a nuclear presentation of the identity mapping

$$(E^\infty, q_{k,K}) \to (E^\infty, q) \, .$$

$\square$

Corollary 3.7. *Let (π, E) be an SF-globalization of a Harish-Chandra module V. Then:*
  (i) *E is a nuclear Fréchet space.*
  (ii) *The topology on E is determined by a countable family of G-continuous Hilbert semi-norms.*

Proof. The proposition above implies that E is nuclear. Thus its topology is induced by a family of Hilbert semi-norms, say $(q_n)_{n \in \mathbb{N}}$. For $c > 0$ large enough the Hilbert semi-norms

$$p_n(v) := \left(\int_G q_n(\pi(g)v)^2 e^{-cd(g)} \, dg \right)^{\frac{1}{2}}$$

are defined and G-continuous. As the $(p_n)_{n \in \mathbb{N}}$ define the same topology as the $(q_n)_{n \in \mathbb{N}}$, the proof is complete. $\square$

Lemma 3.8. *Let $(V, \|\cdot\|)$ be a normed vector space of finite dimension n. Then there exists $\lambda_1, \ldots, \lambda_n$ in V^* with $\|\lambda_i\| = 1$ and $v_1, \ldots, v_n \in V$ with $\|v_i\| \leq n$ such that*

$$\mathrm{id}_V = \sum_{i=1}^n v_i \otimes \lambda_i$$

Proof. We prove this by induction on $n = \dim V$. The case $n = 1$ is clear. Suppose that the assertion is true for all normed vector spaces of dimension smaller than n. Let $U \subset V$ be an $n - 1$-dimensional subspace. Then U endowed with the norm of V restricted to U is a normed space. Then

$$\mathrm{id}_U = \sum_{i=1}^{n-1} u_i \otimes \mu_i$$

with $\|u_i\| \leq n - 1$ and normalized $\mu_i \in \widetilde{U}$. According to Hahn-Banach we can extend each $\mu_i \in \widetilde{U}$ to a functional $\lambda_i \in V^*$ such that $\|\lambda_i\| = 1$. Let $v_n \in \bigcap_{i=1}^{n-1} \ker \lambda_i$ be a normalized element. We use Hahn-Banach to find $\lambda_n \in V^*$ normalized such that $\lambda_n(v_n) = 1$. Then

$$\mathrm{id}_V = \sum_{i=1}^{n-1}(u_i - \lambda_n(u_i)v_n) \otimes \lambda_i + v_n \otimes \lambda_n$$

holds. If we define $v_i := u_i - \lambda_n(u_i)v_n$ for $1 \le i \le n-1$ then induction implies $\|v_i\| \le n$. The lemma follows. $\qquad\square$

3.2.4. *The dual of an SF-globalization.* The material in this subsection is not needed in the sequel of this article. However it contains a fact worthwhile which is worthwhile to mention and thematically fits in our discussion.

Let (π, E) be an SF-globalization of a Harish-Chandra module V. In Corollary 3.7 we have shown that E is a nuclear Fréchet space. As nuclear Fréchet spaces are reflexive, it follows that the dual representation (π^*, E^*) of (π, E) exists and that the bi-dual representation (π^{**}, E^{**}) is naturally isomorphic to (π, E).

Even though the dual representation of an SF-globalization exists one has to be more careful with the associated algebra action of $\mathcal{S}(G)$. For any SF-representation (π, E) we recall that the natural action of $\mathcal{S}(G)$,

$$\mathcal{S}(G) \times E \to E, \quad (f, v) \mapsto \Pi(f)v$$

is continuous, i.e. a continuous bilinear map (Proposition 2.12).

On the dual side we obtain a dual action

$$\mathcal{S}(G) \times E^* \to E^*, \quad (f, \lambda) \mapsto \Pi^*(f)\lambda; \ \Pi^*(f)\lambda = \lambda \circ \Pi(f^*)$$

where $f^*(g) = f(g^{-1})$. Even though (π^*, E^*) exists, the dual action Π^* of $\mathcal{S}(G)$ might not be continuous.

Lemma 3.9. *Let (π, E) be an SF-globalization of a Harish-Chandra module V. Then the dual algebra action $\mathcal{S}(G) \times E^* \to E^*$ is separately continuous.*

Proof. As all operators $\Pi^*(f)$ are continuous, the continuity in the second variable is clear. We move on to continuity in the first variable and fix an element $\lambda \in E^*$. We have to show that the linear map $\mathcal{S}(G) \to E^*$, $f \mapsto \Pi^*(f)\lambda$ is continuous, i.e. continuous at $f = 0$. Let V be a neighborhood of 0 in E^* which we can assume to be the polar of a closed and bounded set $A \subset E$. As E is nuclear Fréchet, it follows that A is compact. Now a neighborhood U of 0 in $\mathcal{S}(G)$ satisfies $\Pi^*(f)U \subset V$ if and only if for all $v \in A$ and $f \in U$ the inequality

$$|\lambda(\Pi(f^*)v)| \le 1$$

holds. As λ is continuous and A is compact, the existence of U follows.
$\square$

3.2.5. *Weighted function spaces.* This subsection is about natural realizations of Harish-Chandra modules in weighted function spaces on G. Instead of left-invariant metrics on G we will use henceforth the concept of a norm on G, see [11], Sect. 2.A.2. We fix a faithful representation $\iota : G \to \mathrm{Gl}(n, \mathbb{R})$ and define a *norm* on G by:

$$\|g\| := \mathrm{tr}(\iota(g)\iota(g)^t) + \mathrm{tr}(\iota(g^{-1})\iota(g)^{-t}) \, .$$

The norm satisfies the following properties:

- $\| \cdot \|$ is smooth and $\|g\| = \|g^{-1}\|$ for all $g \in G$.
- $\|g_1 g_2\| \leq \|g_1\| \cdot \|g_2\|$ for all $g_1, g_2 \in G$.
- For $X \in \mathfrak{g}$ semi-simple with $\mathrm{spec}(\mathrm{ad}\, X) \subset \mathbb{R}$ one has $\| \exp(tX)\| \asymp \| \exp(X)\|^t$ for all $t \geq 0$.

The comparison with the distance function $d(g)$ is obtained vial the squeezing inequalities We record that there exists $C_1, C_2 > 0$ such that

$$C_1 d(g) + C_2 \leq \log \|g\| \leq C_2 d(g) + C_2 \qquad (g \in G) \, ,$$

where $C_1, C_2 > 0$.

For $m \geq 0$ we define the weighted Banach-space

$$C(G)_m := \{ f \in C(G) \mid p^m(f) := \sup_{g \in G} \frac{|f(g)|}{\|g\|^m} < \infty \} \, .$$

We view $C(G)_m$ as a module for G under the right regular action R. Note that this action might not be continuous in general (take $m = 0$ and G not compact). From the properties of the norm one redaily shows that

$$p^m(R(g)v) \leq \|g\|^m \cdot p^m(v)$$

for all $g \in G$. Thus the action is locally equicontinuous. It follows that the smooth vectors for this action $C(G)_m^\infty$ define an SF-module for G. Note that $C(G)_m^\infty \subset C^\infty(G)$ as a consequence of the local Sobolev-Lemma.

In the sequel we will also consider finite direct sums $[C_m(G)]^k$ of $C_m(G)$. If $f = (f_1, \ldots, f_k) \in [C_m(G)]^k$, then we topologize $[C_m(G)]^k$ by the norm

$$p^m(f) := \max_{1 \leq j \leq k} p^m(f_j) \, .$$

We consider $[C_m(G)]^k$ as a G-module for the diagonal action of G and note that the corresponding SF-module is $[C_m(G)^\infty]^k$.

Likewise we associate to $m \geq 0$ the weighted Hilbert-space

$$L^2(G)_m := L^2(G, \|g\|^{-m}dg).$$

Note that the right regular action of G on $L^2(G)_m$ defines a Hilbert representation of G. Let us denote by h^m the corresponding Hilbert norm.

We also consider k-fold direct sum $[L^2(G)_m]^k$ with their natural Hilbert structure. The corresponding Hilbert norm will be also denoted by h^m.

Let $k_0 > 0$ be such that $\int_G \|g\|^{-k_0}\, dg < \infty$. Then for all $m \in \mathbb{R}$ one obtains a continuous embedding

$$(3.1) \qquad [C(G)_{(m-k_0)/2}]^k \to [L^2(G)_m]^k$$

or to phrase it equivalently that there exists a constant $C > 0$ such that

$$(3.2) \qquad h^m \leq C \cdot p^{(m-k_0)/2}.$$

To obtain inequalities of the reverse kind we shall employ the Sobolev Lemma on G. It is not hard to show that the derivatives of the norm function $\|\cdot\|$ are bounded by a multiple of $\|\cdot\|$. Hence we obtain constants $C > 0$ and $l_0 \in \mathbb{N}$ with l_0 independent from m such that

$$(3.3) \qquad p^m \leq C \cdot h_{l_0}^{2m}$$

holds on $[L^2(G)_m^\infty]^k$.

If V is a Harish-Chandra module, then a set of generators $v_1, \ldots, v_k$ of V will be called $\mathcal{Z}(\mathfrak{g})$-*stable* if $\mathrm{span}\{v_1, \ldots, v_k\}$ is $\mathcal{Z}(\mathfrak{g})$-invariant.

Let V be a Harish-Chandra module V and fix a set of $\mathcal{Z}(\mathfrak{g})$-stable generators $\xi = \{\xi_1, \ldots, \xi_k\}$ of $\widetilde{V}$. Attached to ξ we consider the G-equivariant embedding

$$\phi_\xi : V^\infty \to [C^\infty(G)]^k, \quad v \mapsto (m_{\xi_1,v}, \ldots, m_{\xi_k,v})$$

with $m_{\xi_i,v}(g) = \xi_i(\pi(g)v)$ the corresponding matrix coefficient.

We claim that $\mathrm{im}\, \phi_\xi$ lies in some $[C_m(G)^\infty]^k$ for m suitably large. In fact choose a Banach globalization (π, E) of V with norm q. Then

$$\max_{1 \leq j \leq k} |m_{\xi_j,v}(g)| \leq C\|g\|^N q(v) \qquad (v \in V^\infty)$$

for suitable constants N and $C > 0$. Hence $\mathrm{im}\, \phi_\xi \subset [C_N(G)^\infty]^k$.

Let now E_N be the closure of $\phi_\xi(V^\infty)$ in $[C_N(G)]^k$.

Lemma 3.10. *With the notation from above, E_N defines a Banach globalization of V. Moreover, the topology on the smooth vectors E_N^∞ is induced by the K-Sobolev norms $(p_{K,n}^N)_{m \in \mathbb{N}}$.*

Proof. It is clear that E_N is a Banach space. With regard to the norm on E_N the operators $\pi(g)$ are bounded by $\|g\|^N$. Hence the action is locally equicontinuous. Further, as p_N is dominated by q on V^∞ we conclude that all orbit maps $\gamma_v : G \to E_N$ are continuous. Thus $G \times E_N \to E_N$ is a representation by Lemma 2.1.

The final statement is obtained from Lemma 2.5 and the fact that $\mathcal{C}$ leaves ξ stable (see the proof of Lemma 3.5). $\qquad\square$

From our construction it is clear that the smooth vectors E_N^∞ for E_N coincide with the SF-closure $\phi_\xi(V^\infty)$ in $[C_N(G)^\infty]^k$. Let us denote the restriction of p^N to E_N by the same symbol.

Set $N' := 2N + k_0$. Then there is a natural G-equivariant embedding

$$\psi_\xi : V^\infty \to [L^2(G)_{N'}]^k, \quad v \mapsto (m_{\xi_1,v}, \ldots, m_{\xi_k,v})$$

and the closure of the image defines a Hilbert globalization $\mathcal{H}_{N'}$ of V. The important result in this context then is:

Lemma 3.11. *The SF-structure on $\mathcal{H}_{N'}^\infty$ is induced by the K-Sobolev norms $(h_{2l,K}^{N'})_{l \in \mathbb{N}_0}$. To be more precise, for every $l \in \mathbb{N}_0$ the Sobolev norms $h_{2l}^{N'}$ and $h_{2l,K}^{N'}$ are equivalent.*

Proof. Same as in Lemma 3.10. $\qquad\square$

Let us choose the constant l_0 to be even. Then, if we combine the inequalities (3.2), (3.3) with the above Lemma, we obtain for every $m \geq N_0 + k_0/2$ and $l \in \mathbb{N}_0$ a constant $C > 0$ such that

$$(3.4) \qquad\qquad p_{2l}^m(v) \leq C \cdot p_{l_0+2l,K}^{m-k_0/2}(v) \qquad (v \in V)$$

4. Minimal and maximal SF-globalizations of Harish-Chandra modules

Let us introduce a preliminary notion and call a Harish-Chandra module V *good* if it admits a unique SF-globalization. Eventually it will turn out that all Harish-Chandra modules are good (Casselman-Wallach).

As we will see below there are two natural extremal SF-globalizations of a Harish-Chandra module, namely *minimal* and *maximal* SF-globalizations. Eventually they will coincide but they are useful objects towards a proof of the Casselman-Wallach theorem.

4.1. **Minimal globalizations**

An SF-globalization, say V^∞, of an Harish-Chandra module V will be called *minimal* if the following universal property holds: if (π, E) is an SF-globalization of V, then there exists a continuous G-equivariant map $V^\infty \to E$ which extends the identity morphism $V \to V$.

It is clear that minimal globalizations are unique. Let us show that they actually exist. We need to collect some facts about matrix coefficients. In this context we record the following result (see [3]):

Lemma 4.1. *Let (π, E) be a Banach globalization of a Harish-Chandra module V. Then for all $\xi \in \widetilde{V} \subset E^*$ and $v \in V$, the matrix coefficient*

$$m_{\xi,v}(g) = \xi(\pi(g)v) \qquad (g \in G)$$

is an analytic function on G. In particular $m_{\xi,v}$ is independent of the particular Banach globalization (π, E) of V.

We will now give the construction of the minimal globalization of a Harish-Chandra module V. For that let us fix a Banach globalization (π, E) of V. Let $\mathbf{v} = \{v_1, \ldots, v_k\}$ be a set of generators of V and consider the map

$$\mathcal{S}(G)^k \to E, \quad \mathbf{f} = (f_1, \ldots, f_k) \mapsto \sum_{j=1}^{k} \Pi(f_j)v_j \,.$$

This map is linear, continuous and G-equivariant (with $\mathcal{S}(G)^k$ considered as a module for G under the left regular representation). Let us write

$$\mathcal{S}(G)_\mathbf{v} := \{\mathbf{f} \in \mathcal{S}(G)^k \mid \sum_{j=1}^{k} \Pi(f_j)v_j = 0\}$$

for the kernel of this linear map. Note that $\mathcal{S}(G)_\mathbf{v}$ is a closed G-submodule of $\mathcal{S}(G)^k$. We claim that $\mathcal{S}(G)_\mathbf{v}$ is independent of the choice of the particular globalization (π, E) of V: In fact, for $v \in V$ and $f \in \mathcal{S}(G)$ we have $\Pi(f)v = 0$ if and only if $\xi(\Pi(f)v) = 0$ for all $\xi \in \widetilde{V}$. As $g \mapsto m_{\xi,v}(g) = \xi(\pi(g)v)$ is analytic and hence independent of π (Lemma 4.1), the claim follows.

Lemma 2.9 shows that $\mathcal{S}(G)^k/\mathcal{S}(G)_\mathbf{v}$ is an SF-module for G. Since $\Pi(\mathcal{S}(G)^{K \times K})V = V$ for $\mathcal{S}(G)^{K \times K}$ the $K \times K$-finite functions of $\mathcal{S}(G)$, it follows that $\mathcal{S}(G)^k/\mathcal{S}(G)_\mathbf{v}$ is an SF-globalization of V. By construction $\mathcal{S}(G)/\mathcal{S}(G)_\mathbf{v}$ is the minimal globalization V^∞.

We record the following general Lemma on quotients of Harish-Chandra modules in relation to minimal globalizations.

Lemma 4.2. *Let V be a Harish-Chandra module and V^∞ its unique minimal SF-globalization. Let $W \subset V$ be a submodule and $U := V/W$. Let $\overline{W}$ be the closure of W in V^∞. Then $U^\infty = V^\infty/\overline{W}$.*

Proof. Let us write $(\pi_U, V^\infty/\overline{W})$ for the quotient representation obtained from (π, V^∞). Then $\Pi(\mathcal{S}(G))V = V^\infty$ implies that $\Pi_U(\mathcal{S}(G))U = V^\infty/\overline{W}$ and hence the assertion. $\qquad\square$

4.2. Maximal Globalizations

Let us call an SF-globalization of V, say $V^\infty_{\max}$, *maximal* if for any SF-globalization (π, E) of V there exists a continuous linear G-map $E \to V^\infty_{\max}$ sitting above the identity morphism $V \to V$.

It is clear that maximal globalizations are unique provided that they exist. Moreover, in case a maximal globalizations of a Harish-Chandra module V exists, then V is good if and only if $V^\infty = V^\infty_{\max}$.

We will define maximal globalizations using minimal globalizations and duality. For that let $(q^n)_{n\in\mathbb{N}}$ be a family of G-continuous Hilbert semi-norms of $\widetilde{V}^\infty$. We write $(q^n)_l$ for the l-th Sobolev semi-norm of q. We may and will assume that

$$(q^m)_l \in \{q^n \mid n \in \mathbb{N}\}$$

for all m, l.

We define $V^\infty_{\max}$ as the completion of V under the family of the Hilbert-norms $\left(((q^n)^*)_{2l}\right)_{n,l\in\mathbb{N}}$. In order to see that $V^\infty_{\max}$ is the maximal globalization it suffices to prove the following: If (π, E) is Hilbert globalization of V with norm p, then there exists for all m, l and k and a constant $C > 0$ such that

$$(4.1) \qquad ((q^n)^*)_{2l}(v) \leq C \cdot p_{2k}(v) \qquad (v \in V).$$

This is easy to see once we have defined Sobolev norms for sufficiently negative numbers. Let $s < 0$ and $\lambda > 0$. By the functional calculus for the Laplacian (see [4] and specifically [7]) we obtain that $(\lambda\mathbf{1} - \Delta)^s$ is represented by a smooth function on G, say $f_{s,\lambda}$. Moreover given $N > 0$ there we can choose $\lambda > 0$ large enough such that $|f_{s,\lambda}(g)| << \|g\|^{-N}$. Of course using $(\lambda\mathbf{1}-\Delta)^k$ instead of $(\mathbf{1}-\Delta)^k$ in the definition of Sobolev norms does not make a difference. Hence for $s < 0$ small enough p_s is defined for any G-continuous norm p on a Harish-Chandra module. Further we obtain that

$$(p_{2k})^*(\xi) = (p^*)_{-2k}(\xi) \qquad (k \in \mathbb{N}, \xi \in \widetilde{V}).$$

Let us now verify (5.4). As $(q^n)_{n \in \mathbb{N}}$ define the topology for the minimal globalization of $\widetilde{V}$, it follows that there exists a constant $C_1 > 0$ such that

$$(p^*)_{2l}(\xi) \leq C_1 \cdot q^n(\xi) \qquad (\xi \in \widetilde{V}) \, .$$

By duality we thus get that

$$(q^n)^*(v) \leq \frac{1}{C_1}((p^*)_{2l})^*(v) = \frac{1}{C_1} p_{-2l}(v) \qquad (v \in V) \, .$$

As $(p_{-2l})_{2l} = p$ we see that (5.4) follows with $C = \frac{1}{C_1}$.

An immediate consequence of the construction of $V^\infty_{\max}$ is the following basic fact.

Proposition 4.3. *Let V be a Harish-Chandra module. Then V is good if and only if $\widetilde{V}$ is good.*

Likewise the construction of $V^\infty_{\max}$ implies immediately that:

Lemma 4.4. *Let U be a good Harish-Chandra module and U^∞ its unique SF-globalization. Let $V \subset U$ be a submodule and let $\overline{V}$ be the closure of V in U^∞. Then $V^\infty_{\max} = \overline{V}$.*

We conclude this paragraph with an observation which will be frequently used later on.

Lemma 4.5. *Let $V_1 \subset V_2 \subset V_3$ be an inclusion chain of Harish-Chandra modules. Suppose that V_2 and and V_3/V_1 are good. Then V_2/V_1 is good.*

Proof. We first show that V_2/V_1 is good. Let $\overline{V}_3$ be an SF-globalization of V_3. Let $\overline{V}_1$, $\overline{V}_2$ be the closures of $V_{1,2}$ in $\overline{V}_3$. As V_2 is good we have $\overline{V}_2 = V_2^\infty$ and thus Lemma 4.2 implies that $\overline{V_2}/\overline{V_1} = (V_2/V_1)^\infty$. Our second assumption gives $(V_3/V_1)^\infty = \overline{V_3}/\overline{V_1}$ and Lemma 4.4 yields in addition that $\overline{V_2}/\overline{V_1} = (V_2/V_1)^\infty_{\max}$. $\qquad\square$

5. Lower bounds for matrix coefficients

The objective of this section is to show that Harish-Chandra modules are good if and only if they feature certain lower bounds for matrix coefficients which are uniform in the K-type.

Proposition 5.1. *Suppose that V is a good Harish-Chandra module. Let $\xi_1, \ldots, \xi_k$ be a $\mathcal{Z}(\mathfrak{g})$-stable set of generators of $\widetilde{V}$. Then for all G-continuous norms q on V^∞, there exists constants $c_1, c_2, c_3 > 0$ such*

that for all $\tau \in \hat{K}$ and $v \in V[\tau]$ there exist a $g_\tau \in G$ such that $\|g_\tau\| \leq (1 + |\tau|)^{c_1}$ and

$$\max_{1 \leq j \leq k} |\xi_j(\pi(g_\tau)v)| \geq \frac{c_2}{(1 + |\tau|)^{c_3}} \cdot q(v) \,.$$

Proof. [2] By assumption there exists an $n \in \mathbb{N}$ and $C > 0$ such that

$$|\xi_j(\pi(g)v)| \leq C \cdot \|g\|^n q(v)$$

for all $v \in V^\infty$, $g \in G$ and $1 \leq j \leq k$. For $N \geq n$ we write E_N for the Banach completion of V^∞ with respect to the norm

$$p^N(v) := \max_{1 \leq j \leq k} \sup_{g \in G} \frac{|\xi_j(\pi(g)v)|}{\|g\|^N} \qquad (v \in V) \,.$$

We recall that E_N is a Banach module for G (cf. Lemma 3.10).

As V is good, we obtain that

$$(5.1) \qquad\qquad V^\infty = E_N^\infty = E_{N'}^\infty$$

for all $N, N' \geq n$. Now fix N and let $N' = N + l > N$. In view of Lemma 3.10 there exists an $s \in 2\mathbb{N}_0$ and $C > 0$ such that

$$(5.2) \qquad\qquad p^N(v) \leq C \cdot p_{s,K}^{N'}(v)$$

for all $v \in V^\infty$.

Let us fix $\tau \in \hat{K}$, $v \in V[\tau]$ and $g_\tau \in G$ such that $g \mapsto \max_{1 \leq j \leq k} \frac{|\xi_j(\pi(g)v)|}{\|g\|^{N'}}$ becomes maximal at g_τ. We then derive from (5.2) that

$$\max_{1 \leq j \leq k} \frac{|\xi_j(\pi(g_\tau)v)|}{\|g_\tau\|^N} \leq C \cdot (1 + |\tau + \rho_{\mathfrak{k}}|^2 - |\rho_{\mathfrak{k}}|^2)^{\frac{s}{2}} \cdot \max_{1 \leq j \leq k} \frac{|\xi_j(\pi(g_\tau)v)|}{\|g_\tau\|^{N+l}}$$

for all $v \in V[\tau]$, i.e.

$$\|g_\tau\| \leq C \cdot (1 + |\tau + \rho_{\mathfrak{k}}|^2 - |\rho_{\mathfrak{k}}|^2)^{\frac{s}{2l}} \,.$$

Here $\rho_{\mathfrak{k}} \in i\mathfrak{t}^*$ is the usual half sum $\rho_{\mathfrak{k}} = \frac{1}{2} \operatorname{tr} \operatorname{ad}_{\mathfrak{k}}$.

On the other hand (5.1) combined Lemma 3.10 implies likewise that there exists $C > 0$ and $s' > 0$ such that

$$q(v) \leq C \cdot p_{s',K}^{N'}(v)$$

for all $v \in V^\infty$. For $v \in V[\tau]$ we then get

---

[2]Throughout this paper we use the convention that capital constants $C > 0$ might vary from line to line

$$|\xi(\pi(g_\tau)v)| \geq \frac{C \cdot \|g_\tau\|^{N'}}{(1 + |\tau + \rho_{\mathfrak{k}}|^2 - |\rho_{\mathfrak{k}}|^2)^{\frac{s'}{2}}} \cdot q(v).$$

As $\|g\| \geq 1$ for all $g \in G$, all assertions follow. $\qquad\square$

For later reference we record the following converse of the lower bound in the proposition above.

Lemma 5.2. *Let V be a Harish-Chandra module and q be a G-continuous K-invariant Hilbert norm on V^∞. Suppose that there exists a $\mathcal{Z}(\mathfrak{g})$-stable set of generators $\xi_1, \ldots, \xi_k$ of $\widetilde{V}$, constants $c_1, c_2 > 0$ such that for all $v \in V[\tau]$ there exists an $g_\tau \in G$ with $\|g_\tau\| \leq (1 + |\tau|)^{c_1}$ and*

$$\max_{1 \leq j \leq k} |\xi_j(\pi(g_\tau)v)| \geq \frac{1}{(1 + |\tau|)^{c_2}} \cdot q(v).$$

Suppose in addition that the same holds for the dual representation dual norm q^*, i.e. there is $\mathcal{Z}(\mathfrak{g})$-stable set of generators $\xi_1^*, \ldots, \xi_k^*$ of V, constants $c_1', c_2' > 0$ such that for all $v^* \in \widetilde{V}[\tau]$ there exists an $g_\tau' \in G$ with $\|g_\tau'\| \leq (1 + |\tau|)^{c_1'}$ and

$$\max_{1 \leq j \leq k} |\xi_j^*(\pi^*(g_\tau')v^*)| \geq \frac{1}{(1 + |\tau|)^{c_2'}} \cdot q^*(v^*).$$

Then V is good.

Proof. It is sufficient to show the following assertion: If ρ is a G-continuous norm on V^∞, then ρ can be squeezed between two K-Sobolev norms $q_{s,K}$ of q.

In this proof only K-Sobolev norms of q will occur and to simplify notation we will subsequently write q_l instead of $q_{l,K}$.

We first claim that there exists constants $C > 0$ and $l < 0$ such that

(5.3) $$\rho(v) \geq C \cdot q_l(v) \qquad (v \in V).$$

By assumption, there exists $C > 0$ and $s < 0$ such that

(5.4) $$\max_{1 \leq j \leq k} \sup_{\substack{g \in G \\ \|g\| \leq (1+|\tau|)^{c_1}}} |\xi_j(\pi(g)v)| \geq C \cdot q_s(v)$$

for all $v \in V[\tau]$. In order to proceed we have to recall a few facts from Subsection 3.2.5. There exists an $N_0 > 0$ such that for all $N \geq N_0$ the description

$$p^N(v) := \max_{1 \leq j \leq k} \sup_{g \in G} \frac{|\xi_j(\pi(g)v)|}{\|g\|^N}$$

defines a norm on V such that the completion of (V, p^N) yields a Banach representation of G. In view of (5.4) there exists for all $N \geq N_0$ constants C and $l < 0$ such that

$$(5.5) \qquad\qquad p^N(v) \geq C \cdot q_l(v)$$

for all $v \in V[\tau]$. We assert that for all sufficiently large N we can raise $|l|$ such that the above inequality holds for all $v \in V$. If p^N were a K-invariant Hilbert semi-norm, then this assertion would be immediate. Alternatively it would be sufficient to exhibit a K-invariant Hilbert-norm h on V such that a fixed K-Sobolev norm of h dominates some $p^{N'}$ with $N' \geq N_0$ and that h in turn is dominated by p^N. Provided N is sufficiently large, the inequalities (3.2), (3.3) combined with Lemma 3.11 yield that one can find such a Hilbert semi-norm: for appropriate n_0 such a semi-norm is given by

$$h(v)^2 := \sum_{j=1}^{k} \int_G |\xi_j(\pi(g)v)|^2 \cdot \|g\|^{-n_0} \, dg \, .$$

Our assertion, namely $p^N \geq C \cdot q_l$, follows.

By the continuity of ρ there exists an $N'' \in \mathbb{N}$ such that

$$|\xi(\widetilde{\pi}(g)v)| \leq \rho^*(\xi)\rho(\pi(g)v) \leq \rho^*(\xi) \cdot \rho(v) \cdot \|g\|^{N''}$$

for all $g \in G$. Hence for N large enough, p^N is dominated by ρ and the claim is proved.

In order to prove the converse domination of ρ by some q_k we consider the Banach completion E of V by ρ and write (π, E) for the corresponding Banach representation of G. As E^∞ is nuclear we find a continuous Hilbert norm h on E^∞ with $\rho \leq h$. Then choose a Sobolev norm $p := \rho_l$ such that $C \cdot p \geq h$ for some constant $C > 0$. Write F for the Banach completion of V with respect to p and $(\widetilde{\pi}, F)$ for the corresponding Banach representation. Let (π^*, E_c^*) and $(\widetilde{\pi}^*, F_c^*)$ be the continuous duals of (π, E) and $(\widetilde{\pi}, F)$.

We apply our claim to the dual Harish-Chandra module $\widetilde{V}$ and the Banach globalization $(\widetilde{\pi}, F_c^*)$ of $\widetilde{V}$ (see Lemma3.3). Thus we get constants $l' < 0$ and $C > 0$ such that $p^* \geq C \cdot q_{l'}^*$ on $\widetilde{V}$. Hence $h^* \geq C \cdot q_s^*$ on $\widetilde{V}$. But as both h^* and q_s^* are Hilbert norms reflexivity implies that $h \leq C \cdot q_{-l'}$ on V. But then $\rho \leq C \cdot q_{-l'}$ and the proof of the Lemma is complete. $\qquad\square$

6. Discrete series

The objective of this section is to show that every Harish-Chandra module belonging to the discrete series is good.

Let $Z < G$ be the center of G. Throughout this section V shall denote a unitarizable irreducible Harish-Chandra module, i.e. there exists a unitary irreducible globalization $(\pi, \mathcal{H})$ of V. We say that V is *square integrable* or *belongs to the discrete series* if for all $v \in V$ and $\xi \in \widetilde{V}$ one has

$$\int_{G/Z} |m_{\xi,v}(g)|^2 \, d(gZ) < \infty.$$

In this situation, there exists a constant $d(\pi)$, the formal degree, such that for every unitary norm p on V one has

$$\frac{1}{d(\pi)} p(v)^2 p^*(\xi)^2 = \int_{G/Z} |m_{\xi,v}(g)|^2 \, d(gZ) \qquad (v \in V, \xi \in \widetilde{V}).$$

Proposition 6.1. *Let V be a Harish-Chandra module of the discrete series. Then V is good.*

Proof. We first show that the smooth vectors $\mathcal{H}^\infty$ of the unitary globalization $(\pi, \mathcal{H})$ coincide with the maximal globalization.

Let us fix a G-continuous Hilbert-norm q on V. Then there exists constants $C > 0$ and $N \in \mathbb{N}$ such that
(6.1)
$$\int_{G/Z} |m_{\xi,v}(g)|^2 \frac{1}{\|\operatorname{Ad}(g)\|^N} \, d(gZ) \leq C q(v)^2 q^*(\xi)^2 \qquad (v \in V, \xi \in \widetilde{V}).$$

We let $\xi_1, \xi_2, \ldots$ be an orthonormal basis of $\widetilde{V}$ with respect to p^* and with the following properties:

- Each ξ_k belongs to a K-type $\widetilde{V}[\tau_k]$ for some $\tau_k \in \hat{K}$,
- $|\tau_n| \leq |\tau_m|$ for all $n \leq m$.

For every $n \in \mathbb{N}$ we define a G-continuous Hilbert-norm on V by

$$\rho^n(v)^2 := \int_{G/Z} |m_{\xi_n,v}(g)|^2 \frac{1}{\|\operatorname{Ad}(g)\|^N} \, d(gZ) \qquad (v \in V)$$

Observe that $\rho^n(v)^2 \leq \frac{1}{d(\pi)} p(v)^2$ for all $v \in V$. We claim that the SF-structure of $\mathcal{H}^\infty$ and the SF-structure defined by the Sobolev-norms of the ρ^n's coincide. In order to see that we choose $c > 0$ large enough such that $\sum_{n=1}^\infty \frac{1}{(1+|\tau_n|)^c} < \infty$.

For all $v \in V$ we first note that

$$p(v)^2 = \sum_{n=1}^{\infty} |\xi_n(v)|^2 \, .$$

By the local Sobolev-Lemma (applied to a Ball arround $\mathbf{1}$ in G) we find a $k > c$ and $C > 0$ such that

$$\rho^n(v)^2 = \int_{G/Z} |m_{\xi_n,v}(g)|^2 \frac{1}{\|\operatorname{Ad}(g)\|^N} \, d(gZ) \geq C \frac{1}{(1+|\tau_n|)^k} |\xi_n(v)|^2$$

for all $n \in \mathbb{N}$ (use Lemma 3.5). This proves the claim. In view of (6.1), we get that $\mathcal{H}^{\infty} = V_{\max}^{\infty}$.

Finally, the fact that V belong to the discrete series implies (in fact is equivalent to the fact) that $\widetilde{V}$ belong to the discrete series, in particular is unitarizable. Thus maximal and minimal globalization coincide and the proof is complete.

$\square$

7. Spherical principal series representation

This section is devoted to a thorough study of spherical principal series representation of G. We will introduce a Dirac-type sequence for such representations and establish lower bounds for matrix-coefficients which are uniform in the K-types. These lower bounds are essentially sharp, locally uniform in the representation parameter, and stronger than the more abstract estimates in Proposition 5.1.

The lower bounds established give us a constructive method for finding Schwartz-functions representing a given smooth vector.

Let us write $G = NAK$ for an Iwasawa decomposition of G. Accordingly we decompose elements $g \in G$ as

$$g = \widetilde{n}(g)\widetilde{a}(g)\widetilde{k}(g)$$

with $\widetilde{n}(g) \in N$, $\widetilde{a}(g) \in A$ and $\widetilde{k}(g) \in K$. Set $M = Z_K(A)$ and define a minimal parabolic subgroup of G by $P_{\min} = NAM$.

The Lie algebras of A, N and K shall be denoted by $\mathfrak{a}, \mathfrak{n}$ and $\mathfrak{k}$. Complexification of Lie-algebras are indicated with a $\mathbb{C}$-subscript, i.e. $\mathfrak{g}_{\mathbb{C}}$ is the complexification of $\mathfrak{g}$ etc. As usually we define $\rho \in \mathfrak{a}^*$ by $\rho(Y) := \frac{1}{2}\operatorname{tr}(\operatorname{ad}_{\mathfrak{n}} Y)$ for $Y \in \mathfrak{a}$.

The smooth spherical principal series with parameter $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is defined by

$$\mathcal{H}_\lambda^\infty := \{f \in C^\infty(G) \mid (\forall nam \in P_{\min}, \forall g \in G)$$
$$f(namg) = a^{\rho+\lambda} f(g)\}$$

We note that R defines a smooth representation of G on $\mathcal{H}_\lambda^\infty$ which we denote henceforth by π_λ. The restriction map to K defines a K-isomorphism:

$$\mathrm{Res}_K : \mathcal{H}_\lambda^\infty \to C^\infty(K\backslash M), \quad f \mapsto f|_K.$$

The resulting action of G on $C^\infty(M\backslash K)$ is given by

$$[\pi_\lambda(g)f](Mk) = f(M\widetilde{k}(kg))\widetilde{a}(kg)^{\lambda+\rho}.$$

This action lifts to a continuous action on the Hilbert completion $\mathcal{H}_\lambda = L^2(M\backslash K)$ of $C^\infty(M\backslash K)$. We note that this representation is unitary provided that $\lambda \in i\mathfrak{a}^*$.

We denote by V_λ the K-finite vectors of π_λ and note that $V_\lambda = \mathbb{C}[M\backslash K]$ as K-module. For later reference we record that the dual representation of $(\pi_\lambda, \mathcal{H}_\lambda)$ is isomorphic to $(\pi_{-\lambda}, \mathcal{H}_{-\lambda})$ via the G-equivariant pairing

$$(7.1) \quad (\cdot,\cdot): \ \mathcal{H}_{-\lambda} \times \mathcal{H}_\lambda \to \mathbb{C}, \quad (\xi,v) := \int_{M\backslash K} \xi(Mk)v(Mk) \ d(Mk).$$

Here G-equivariance means that

$$(\pi_{-\lambda}(g)\xi, v) = (\xi, \pi_\lambda(g^{-1})v)$$

for all $g \in G$.

7.1. K-expansion of smooth vectors

We recall $\hat{K}$, the set of equivalence classes of irreducible unitary representations of K. If $[\tau] \in \hat{K}$ we let (τ, U_τ) be a representative. Further we write $\hat{K}_M$ for the subset of M-spherical equivalence classes, i.e.

$$[\tau] \in \hat{K}_M \iff U_\tau^M := \{u \in U_\tau \mid \tau(m)u = u \ \forall m \in M\} \neq \{0\}.$$

Given a finite dimensional representation (τ, U_τ) of K we denote by (τ^*, U_τ^*) its dual representation. With each $[\tau] \in \hat{K}_M$ comes the realization mapping

$$r_\tau : U_\tau \otimes (U_\tau^*)^M \to L^2(M\backslash K), \quad u \otimes \eta \mapsto (Mk \mapsto \eta(\tau(k)u)).$$

Let us fix a K-invariant inner product on U_τ. This inner product induces a K-invariant inner product on U_τ^*. We obtain an inner product on $U_\tau \otimes (U_\tau^*)^M$ which is independent of the chosen inner product on U_τ. If we denote by $d(\tau)$ the dimension of U_τ, then Schur-orthogonality implies that

$$\frac{1}{d(\tau)} \|u \otimes \eta\|^2 = \|r_\tau(u \otimes \eta)\|^2_{L^2(M \backslash K)}.$$

Taking all realization maps together we arrive at a K-module isomorphism

$$\mathbb{C}[M \backslash K] = \sum_{\tau \in \hat{K}_M} U_\tau \otimes (U_\tau^*)^M.$$

Let us fix a maximal torus $\mathfrak{t} \subset \mathfrak{k}$ and a positive chamber $\mathcal{C} \subset i\mathfrak{t}^*$. We often identify τ with its highest weight in $\mathcal{C}$ and write $|\tau|$ for the norm (with respect to the positive definite form B) of the highest weight. As $d(\tau)$ is polynomial in τ we arrive at the following characterization of the smooth functions:

$$C^\infty(M \backslash K) = \Big\{ \sum_{\tau \in \hat{K}_M} c_\tau u_\tau \mid c_\tau \in \mathbb{C}, u_\tau \in U_\tau \otimes (U_\tau^*)^M, \|u_\tau\| = 1$$

$$(\forall N \in \mathbb{N}) \sum_{\tau \in \hat{K}_M} |c_\tau|(1 + |\tau|)^N < \infty \Big\}.$$

Let us denote by δ_{Me} the point-evaluation of $C^\infty(M \backslash K)$ at the base point Me. We decompose δ_{Me} into K-types:

$$\delta_{Me} = \sum_{\tau \in \hat{K}_M} \delta_\tau$$

where

$$\delta_\tau = d(\tau) \sum_{i=1}^{l(\tau)} u_i \otimes u_i^*$$

with $u_1, \ldots, u_{l(\tau)}$ any basis of U_τ^M and $u_1^*, \ldots, u_{l(\tau)}^*$ its dual basis. For $1 \leq i, j \leq l(\tau)$ we set

$$\delta_\tau^{i,j} := u_i \otimes u_j^*$$

and record that $\delta_\tau = d(\tau) \sum_{i=1}^{l(\tau)} \delta_\tau^{i,i}$. Note the following properties of δ_τ and $\delta_\tau^{i,j}$:

- $\|\delta_\tau^{i,i}\|_\infty = \delta_\tau^{i,i}(Me) = 1$.
- $\delta_\tau * \delta_\tau = \delta_\tau$.
- $\delta_\tau * f = f$ for all $f \in L^2(M \backslash K)_\tau := \operatorname{im} r_\tau$.

7.2. Non-compact model

We have seen that the restriction map Res_K realizes $\mathcal{H}_\lambda^\infty$ as a function space on $M\backslash K$. Another standard realization will be useful for us. Let us denote by $\overline{N}$ the opposite of N. Here, $\mathfrak{n}$ stands for the Lie algebra of N. As $NAM\overline{N}$ is open and dense in G we obtain a faithful restriction mapping:

$$\mathrm{Res}_{\overline{N}} : \mathcal{H}_\lambda^\infty \to C^\infty(\overline{N}), \quad f \mapsto f|_{\overline{N}}.$$

Note that this map is not onto. The transfer of compact to non-compact model is given by

$$\mathrm{Res}_{\overline{N}} \circ \mathrm{Res}_K^{-1} : C^\infty(M\backslash K) \to C^\infty(\overline{N}),$$
$$f \mapsto F; \ F(\overline{n}) := \widetilde{a}(\overline{n})^{\lambda+\rho} f(\widetilde{k}(\overline{n}))$$

The transfer of the Hilbert space structure on $\mathcal{H}_\lambda = L^2(M\backslash K)$ results in the L^2-space $L^2(\overline{N}, \widetilde{a}(\overline{n})^{-2\,\mathrm{Re}\,\lambda} d\overline{n})$ with $d\overline{n}$ an appropriately normalized Haar measure on $\overline{N}$. In the sequel we also write $\mathcal{H}_\lambda$ for $L^2(\overline{N}, \widetilde{a}(\overline{n})^{-2\,\mathrm{Re}\,\lambda} d\overline{n})$ in the understood context. The full action of G in the non-compact model is not of relevance to us, however we will often use the A-action which is much more transparent in the non-compact picture:

$$[\pi_\lambda(a)f](\overline{n}) = a^{\lambda+\rho} f(a^{-1}\overline{n}a)$$

for all $a \in A$ and $f \in L^2(\overline{N}, \widetilde{a}(\overline{n})^{-2\,\mathrm{Re}\,\lambda} d\overline{n})$.

7.3. K-finite vectors with fast decay

The fact that Res_K is an isomorphism follows from the geometric fact that $P_{\min}\backslash G \simeq M\backslash K$. Now $\overline{N}$ embeds into $P_{\min}\backslash G = M\backslash K$ as an open dense subset. In fact the complement is algebraic and we are going to describe it as the zero set of a K-finite functions f on $M\backslash K$. We will show that f can be chosen such that f restricted to $\overline{N}$ has polynomial decay of arbitrary fixed order.

Let (σ, W) be a finite dimensional faithful irreducible representation of G. We assume that W is K-spherical, i.e. W admits a non-zero K-fixed vector, say v_K. It is known that σ is K-spherical if and only if there is a real line $L \subset W$ which is fixed under $\overline{P}_{\min} = MA\overline{N}$. Let $L = \mathbb{R}v_0$ and $\mu \in \mathfrak{a}^*$ be such that $\sigma(a)v_0 = a^\mu \cdot v_0$ for all $a \in A$, in other words: v_0 is a lowest weight vector of σ and μ is the corresponding lowest weight.

Let now $\langle \cdot, \cdot \rangle$ be an inner product on W which is θ-covariant: if $g = k\exp(X)$ for $k \in K$ and $X \in \mathfrak{p}$ and $\theta(g) := k\exp(-X)$, then covariance means

$$\langle \sigma(g)v, w \rangle = \langle v, \sigma(\theta(g)^{-1})w \rangle$$

for all $v, w \in W$ and $g \in G$. Such an inner product is unique up to scalar by Schur's Lemma. Henceforth we request that v_0 is normalized and we fix v_K by $\langle v_0, v_K \rangle = 1$. Consider on G the function

$$f_\sigma(g) := \langle \sigma(g)v_0, v_0 \rangle .$$

The restriction of f_σ to K is also denoted by f_σ.

Let now $\overline{n} \in \overline{N}$ and write $\overline{n} = \widetilde{n}(\overline{n})\widetilde{a}(\overline{n})\widetilde{k}(\overline{n})$ according to the Iwasawa decomposition. Then $\widetilde{k}(\overline{n}) = n^*\widetilde{a}(\overline{n})^{-1}\overline{n}$ for some $n^* \in N$. Consequently

$$f_\sigma(\widetilde{k}(\overline{n})) = \widetilde{a}(\overline{n})^{-\mu} .$$

If $(\overline{n}_j)_j$ is a sequence in $\overline{N}$ such that $\widetilde{k}(\overline{n}_j)$ converges to a point in $M\backslash K - \widetilde{k}(\overline{N}) =: M\backslash K - \overline{N}$, then $\widetilde{a}(\overline{n}_j)^{-\mu} \to 0$. Hence

$$M\backslash K - \overline{N} \subset \{Mk \in M\backslash K \mid f_\sigma(k) = 0\} .$$

As f_σ is non-negative one obtains for all regular σ that equality holds:

$$M\backslash K - \overline{N} = \{Mk \in M\backslash K \mid f_\sigma(k) = 0\}$$

(this reasoning is not new and goes back to Harish-Chandra). Let us fix such a σ now.

We claim that the mapping $\overline{n} \to f_\sigma(\overline{n})$ is the inverse of a polynomial mapping, i.o.w. the map

$$\overline{N} \to \mathbb{R}, \quad \overline{n} \mapsto \widetilde{a}(\overline{n})^\mu$$

is a polynomial map. But this follows from

$$\widetilde{a}(\overline{n})^\mu = \langle \sigma(\overline{n})v_K, v_0 \rangle$$

by means of our normalizations.

In order to make estimates later on we introduce coordinates on $\overline{N}$. For that we first write $\overline{\mathfrak{n}}$ as semi-direct product of $\mathfrak{a}$-root vectors:

$$\overline{\mathfrak{n}} = \mathbb{R}X_1 \ltimes (\mathbb{R}X_2 \ltimes (\ldots \ltimes \mathbb{R}X_n)\ldots) .$$

Accordingly we write elements of $\overline{\mathfrak{n}}$ as $X := \sum_{j=1}^n x_j X_j$ with $x_i \in \mathbb{R}$. We note the following two facts:

- The map

$$\Phi : \overline{\mathfrak{n}} \to \overline{N}, \quad X \mapsto \overline{n}(X) := \exp(x_1 X_1) \cdot \ldots \cdot \exp(x_n X_n)$$

is a diffeomorphism.

- One can normalize the Haar measure $d\overline{n}$ of $\overline{N}$ in such a way that:

$$\Phi^*(d\overline{n}) = dx_1 \cdot \ldots \cdot dx_n \,.$$

We introduce a norm on $\overline{\mathfrak{n}}$ by setting

$$\|X\|^2 := \sum_{j=1}^n |x_j|^2 \qquad (X \in \overline{\mathfrak{n}}) \,.$$

Finally we set

$$f_\sigma(X) := f_\sigma(\widetilde{k}(\overline{n}(X))) = \widetilde{a}(\overline{n}(X))^{-\mu}$$

and summarize our discussion.

Lemma 7.1. *Let $m > 0$. Then there exists $C > 0$ and a finite dimensional K-spherical representation (σ, W) of G such that:*

(i) $M\backslash K - \overline{N} = \{Mk \in M\backslash K \mid f_\sigma(k) = 0\}$.
(ii) $|f_\sigma(X)| \leq C \cdot (1 + \|X\|)^{-m}$ *for all* $X \in \overline{\mathfrak{n}}$.

7.4. **Dirac type sequences**

Dirac sequences do not exist for Hilbert representations as they are features of an L^1-theory. However, rescaled they exist for the Hilbert representations we shall consider.

Recall our function f_σ on $M\backslash K$. We let $\xi = \xi_\sigma$ be the corresponding function transferred to $\overline{N} \simeq \overline{\mathfrak{n}}$ i.e.

$$\xi(X) := \widetilde{a}(\overline{n}(X))^{\rho+\lambda} f_\sigma(\widetilde{k}(\overline{n}(X))) = \widetilde{a}(\overline{n}(X))^{\rho+\lambda-\mu} \,.$$

It is clear that ξ is a K-finite vector for π_λ.

We recall that $\xi(X)$ satisfies the inequality

$$|\xi(X)| \leq C \cdot (1 + \|X\|)^{-m}$$

where we can choose m as large as we wish (provided σ is sufficiently regular and large). Record the normalization $\xi(0) = 1$.

We will chose m at least that large that ξ becomes integrable and write $\|\xi\|_1$ for the corresponding $L^1(\overline{N})$-norm.

The operators $\pi_\lambda(a)$ can be understood as scaling operators in the non-compact picture. For our purpose the scaling in one direction of A will be sufficient. To make this precise we fix an element $Y \in \mathfrak{a}$ such that $\alpha(Y) \geq 1$ for all roots $\alpha \in \Sigma(\mathfrak{a}, \mathfrak{n})$. For $t > 0$ we put

$$a_t := \exp((\log t)Y) \,.$$

Note that for $\eta \in \mathfrak{a}_{\mathbb{C}}^*$ one has

$$a_t^\eta = t^{\eta(Y)} \,.$$

In the sequel we will often abbreviate and simply write t^η for $t^{\eta(Y)}$.

In order to explain the idea of this section let us assume for a moment that λ is real. Then ξ is a positive function and

$$\left(\frac{a_t^{\rho - \lambda}}{\|\xi\|_1} \cdot \pi_\lambda(a_t)\xi \right)_{t > 0}$$

forms a Dirac sequence for $t \to \infty$ (If λ is not real, then ξ is oscillating and we have to be slightly more careful).

In the compact picture this means

$$\lim_{t \to \infty} \frac{a_t^{\rho - \lambda}}{\|\xi\|_1} \cdot \pi_\lambda(a_t)f_\sigma = \delta_{Me} = \sum_{\tau \in \hat{K}_M} \delta_\tau .$$

It is our goal to understand this limit in the K-types: How large do we have to choose t in dependence of τ such that the τ-isotypical part of $\frac{a_t^{\rho - \lambda}}{\|\xi\|_1} \cdot \pi_\lambda(a_t)f_\sigma$ approximates δ_τ well. It turns out that t can be chosen polynomially in τ. If we denote by D_τ the transfer of the character δ_τ to the non-compact model, the precise statement is as follows.

Theorem 7.2. *Let $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and $N > 0$. Then there exists a choice of σ and hence of $\xi = \xi_\sigma \in V_\lambda$, constants $c > 2$, $C > 0$ such that for all $\tau \in \hat{K}_M$ one has*

$$[\pi_\lambda(a_{t(\tau)})\xi]_\tau = a_{t(\tau)}^{-\rho + \lambda} \cdot I_\xi \cdot D_\tau + R_\tau$$

where $t(\tau) := (1 + |\tau|)^c$,

$$I_\xi := \int_{\overline{N}} \xi(\overline{n}) \ d\overline{n} \neq 0$$

and remainder $R_\tau \in \mathcal{H}_\lambda[\tau]$ satisfying

$$\frac{\|R(\tau)\|}{|a_{t(\tau)}^{-\rho + \lambda}|} \leq \frac{C}{(1 + |\tau|)^N} .$$

Proof. Recall the M-fixed functions $\delta_\tau^{i,j} \in L^2(M \backslash K)_\tau$, $1 \leq i, j \leq l(\tau)$ for $\tau \in \hat{K}_M$. In the sequel we abbreviate and set $d := d(\tau)$, $l := l(\tau)$.

Let $D_\tau^{i,j}(\overline{n}) = \widetilde{a}(\overline{n})^{\rho + \lambda} \delta_\tau^{i,j}(\widetilde{k}(\overline{n}))$ the transfer of $\delta_\tau^{i,j}$ to the non-compact model. We also set $D_\tau^{i,j}(X) := D_\tau^{i,j}(\overline{n}(X))$ for $X \in \overline{\mathfrak{n}}$. Let us note that $|D_\tau^{i,j}(0)| = \delta_{ij}$.

As $\pi_\lambda(a)\xi$ is M-fixed for all $a \in A$ we conclude that

$$[\pi_\lambda(a_t)\xi]_\tau = \sum_{i,j=1}^{l} b_{i,j}(t) \cdot d \cdot D_\tau^{ij} .$$

If $\langle\cdot,\cdot\rangle$ denotes the Hermitian bracket on $\mathcal{H}_\lambda = L^2(\overline{N}, \widetilde{a}(\overline{n})^{-2\operatorname{Re}\lambda}d\overline{n})$, then the coefficients $b_{i,j}(t)$ are obtained by the integrals

$$b_{i,j}(t) = \langle\pi_\lambda(a_t)\xi, D_\tau^{i,j}\rangle = \int_{\overline{\mathfrak{n}}}(\pi_\lambda(a_t)\xi)(X)\cdot\overline{D_\tau^{i,j}(X)}\cdot\widetilde{a}(\overline{n}(X))^{-2\operatorname{Re}\lambda}\,dX\,,$$

where we used the notation

$$dX := dx_1\cdot\ldots\cdot dx_n$$

for $X = \sum_{j=1}^n x_j X_j$.

Fix $1 \geq t_0 > 0$ and set $t = t_0^{-2}$.

We split the integrals for $b_{i,j}(t)$ into two parts $b_{i,j}(t) = b_{i,j}^1(t) + b_{i,j}^2(t)$ with

$$b_{i,j}^1(t) := \int_{\{\|X\|\geq t_0\}}(\pi_\lambda(a_t)\xi)(X)\cdot\overline{D_\tau(X)}\cdot\widetilde{a}(\overline{n}(X))^{-2\operatorname{Re}\lambda}\,dX\,.$$

In our first step of the proof we wish to estimate $b_{i,j}^1(t)$. For that let $C, q_1 > 0$ be such that

$$\widetilde{a}(\overline{n}(X))^{-2\operatorname{Re}\lambda} \leq C\cdot(1+\|X\|)^{q_1}\,.$$

Likewise, by the definition of $D_\tau^{i,j}$ we obtain constants $C, q_2 > 0$ which only depend on $\operatorname{Re}\lambda$ and such that

$$|D_\tau^{i,j}(X)| \leq C\cdot(1+\|X\|)^{q_2}$$

for all τ and $1 \leq i, j \leq l$. Set $q := q_1 + q_2$.

From the inequalities just stated we arrive at:

$$|b_{i,j}^1(t)| \leq C\cdot t^{\operatorname{Re}\lambda+\rho}\int_{\{\|X\|\geq t_0\}}|\xi(\operatorname{Ad}(a_t)^{-1}X)|\cdot(1+\|X\|)^q\,dX\,.$$

As $|\xi(X)| \leq C\cdot(1+\|X\|)^{-m}$ for some constants $C, m > 0$ we thus get that

$$|b_1^i(\tau,t)| \leq C\cdot t^{\operatorname{Re}\lambda+\rho}\int_{\{\|X\|\geq t_0\}}\frac{(1+\|X\|)^q}{(1+\|\operatorname{Ad}(a_t)^{-1}X\|)^m}\,dX\,.$$

By the definition of a_t we get that $\|\operatorname{Ad}(a_t)^{-1}X\| \geq t\|X\|$ and hence

$$|b_{i,j}^1(t)| \leq C\cdot t^{\operatorname{Re}\lambda+\rho}\int_{\{\|X\|\geq t_0\}}\frac{(1+\|X\|)^q}{(1+t\|X\|)^m}\,dX\,.$$

We continue this estimate by employing polar coordinates for $X \in \overline{\mathfrak{n}}$:

$$|b_{i,j}^1(t)| \leq C \cdot t^{\mathrm{Re}\,\lambda + \rho} \int_{t_0}^{\infty} \frac{r^n (1+r)^q}{(1+tr)^m} \frac{dr}{r}$$

$$= C \cdot t_0^{n-2(\mathrm{Re}\,\lambda + \rho)} \int_1^{\infty} \frac{r^n (1+t_0 r)^q}{(1+tt_0 r)^m} \frac{dr}{r}$$

$$= C \cdot t_0^{n-2(\mathrm{Re}\,\lambda + \rho)} \int_1^{\infty} \frac{r^n (1+t_0 r)^q}{(1+t_0^{-1} r)^m} \frac{dr}{r}$$

$$= C \cdot t_0^{n-2(\mathrm{Re}\,\lambda + \rho)+m} \int_1^{\infty} \frac{r^n (1+t_0 r)^q}{(t_0 + r)^m} \frac{dr}{r}$$

$$\leq C \cdot t_0^{n-2(\mathrm{Re}\,\lambda + \rho)+m} \int_1^{\infty} r^{n+q-m} \frac{dr}{r}.$$

Henceforth we request that $m > n + q + 1$. Thus for every $m' > 0$ there exist a choice of ξ and a constant $C > 0$ such that

(7.2) $$|b_{i,j}^1(t)| \leq C \cdot t^{-m'}.$$

Next we choose t in relationship to $|\tau|$. Basic finite dimensional representation theory yields that in a fixed compact eighborhood of $X = 0$ the gradient of $D_\tau^{i,j}$ is bounded by $C \cdot (1 + |\tau|)$ for a constant C independent of τ. Let $\gamma > 1$. Then for $\|X\| \leq (1 + |\tau|)^{-\gamma}$ the mean value theorem yields the following estimate

(7.3) $$|D_\tau^{i,j}(X) - D_\tau^{i,j}(0)| \leq C \cdot (1 + |\tau|)^{-\gamma+1}$$

This brings us to our choice of t, namely

$$t = t(\tau) := (1 + |\tau|)^{2\gamma}.$$

Recall the definition of

$$I_\xi = \int_{\overline{\mathfrak{n}}} \xi(X) \, dX$$

Here we might face the obstacle that I_ξ might be zero. However as $\xi(X) = \widetilde{a}(n(X))^{\rho + \lambda - \mu}$, there are for each μ in the "half line" $\mathbb{N}\mu$ infinitely many lowest weights for which $I \neq 0$ (apply Carleman's theorem, see [10], 3.71). So for any m' we find such a non-zero I_ξ.

In the following computation we will use the simple identity:

$$\int_{\overline{\mathfrak{n}}} \pi_\lambda(a_t) f(X) \, dX = t^{\lambda - \rho} \int_{\overline{\mathfrak{n}}} f(X) \, dX$$

for all integrable functions f. Now if $i \neq j$, then $D_\tau^{i,j}(0) = 0$ and we obtain from (7.2) and (7.3) that

$$b_{i,j}(t) = \int_{\{\|X\| \leq t_0\}} (\pi_\lambda(a_t)\xi)(X) \cdot \overline{D_\tau^{i,j}(X)}\ \widetilde{a}(\overline{n}(X))^{-2\operatorname{Re}\lambda} dX$$

$$+ O\left(\frac{1}{(1+|\tau|)^{2\gamma m'}}\right)$$

$$= \int_{\{\|X\| \leq t_0\}} (\pi_\lambda(a_t)\xi)(X)\overline{(D_\tau^{i,j}(X) - D_\tau^{i,j}(0))}\ \widetilde{a}(\overline{n}(X))^{-2\operatorname{Re}\lambda}\ dX$$

$$+ O\left(\frac{1}{(1+|\tau|)^{2\gamma m'}}\right)$$

$$= t^{\lambda-\rho} \cdot \|\xi\|_1 \cdot O\left(\frac{1}{(1+|\tau|)^{\gamma-1}}\right) + O\left(\frac{1}{(1+|\tau|)^{2\gamma m'}}\right).$$

For $i = j$ we have $D_\tau^{i,i}(0) = 1$ and we obtain in a similar fashion that

$$b_{i,i}(t) = \int_{\{\|X\| \leq t_0\}} (\pi_\lambda(a_t)\xi)(X) \cdot \widetilde{a}(\overline{n}(X))^{-2\operatorname{Re}\lambda}\ dX$$

$$+ t^{\lambda-\rho} \cdot \|\xi\|_1 \cdot O\left(\frac{1}{(1+|\tau|)^{\gamma-1}}\right) + O\left(\frac{1}{(1+|\tau|)^{2\gamma m'}}\right)$$

$$= t^{\lambda-\rho} \cdot I_\xi + O\left(\frac{1}{(1+|\tau|)^{\gamma-1}}\right) + O\left(\frac{1}{(1+|\tau|)^{2\gamma m'}}\right).$$

If we choose $c := 2\gamma$ and $\gamma - 1 = N$ and m' large enough, the assertion of the theorem follows. $\qquad\square$

The proof of the theorem shows that the approximation can be made uniformly on any compact subset $Q \subset \mathfrak{a}_\mathbb{C}^*$. We further observe that $a_{t(\tau)}$ is bounded from above and below by powers of $1 + |\tau|$. If switch to the compact models $\mathcal{H}_\lambda = L^2(M\backslash K)$ and denote f_σ also by ξ, then an alternative version of the theorem is as follows:

Theorem 7.3. *Let $Q \subset \mathfrak{a}_\mathbb{C}^*$ be a compact subset and $N > 0$. Then there exists $\xi \in \mathbb{C}[M\backslash K]$ and constants $c_1, c_2 > 0$ such that for all $\tau \in \hat{K}_M$, $\lambda \in Q$, there exists $a_\tau \in A$, independent of λ, with $\|a_\tau\| \leq (1 + |\tau|)^{c_1}$ and numbers $b(\lambda, \tau) \in \mathbb{C}$ such that*

$$\|[\pi_\lambda(a_\tau)\xi]_\tau - b(\lambda,\tau)\delta_\tau\| \leq \frac{1}{(|\tau|+1)^{N+c_2}}$$

and

$$|b(\lambda,\tau)| \geq \frac{1}{(1+|\tau|)^{c_2}}.$$

Here $\| \cdot \|$ refers to the norm in $L^2(M\backslash K)$.

Finally we deduce the following lower bound for matrix coefficients. Recall the non-degenerate complex bilinear G-equivariant pairing $(\cdot, \cdot)$ between $\mathcal{H}_\lambda$ and $\mathcal{H}_{-\lambda}$.

Corollary 7.4. *Let $Q \subset \mathfrak{a}^*_\mathbb{C}$ be a compact subset. Then there exists $\xi \in \mathbb{C}[M\backslash K]$, constants $c_1, c_2, c_3 > 0$ such that*

$$\sup_{\substack{g \in G \\ \|g\| \leq (1+|\tau|)^{c_1}}} |(\pi_\lambda(g)\xi, v)| \geq c_2 \frac{1}{(1+|\tau|)^{c_3}} \|v\|$$

for all $\lambda \in Q$, $\tau \in \hat{K}_M$ and $v \in V_{-\lambda}[\tau]$. Here $\|v\|$ refers to the norm on $\mathcal{H}_{-\lambda} = L^2(M\backslash K)$. In particular there exist a $s \in \mathbb{R}$ such that

$$\sup_{\substack{g \in G \\ \|g\| \leq (1+|\tau|)^{c_1}}} |(\pi_\lambda(g)\xi, v) \geq c_2 \|v\|_{s,K}$$

for all $\lambda \in Q$, $\tau \in \hat{K}_M$ and $v \in V_{-\lambda}[\tau]$.

Note $V_\lambda^* \simeq V_{-\lambda}$. Thus Lemma 5.2 in conjunction with the above Corollary yields the Casselman-Wallach Theorem for spherical principal series:

Corollary 7.5. *Let $\lambda \in \mathfrak{a}^*_\mathbb{C}$ and V_λ the Harish-Chandra module of the corresponding spherical principal series. Then V_λ admits a unique smooth Fréchet globalization.*

7.5. **Constructions in the Schwartz algebra**

Let us fix a relatively compact open neighborhood $Q \subset \mathfrak{a}^*_\mathbb{C}$. We choose the K-finite element $\xi \in \mathbb{C}[M\backslash K]$ such that the conclusion of Theorem 7.3 is satisfied.

Lemma 7.6. *Let U be an $\mathrm{Ad}(K)$-invariant neighborhood of $\mathbf{1}$ in G and $\mathcal{F}(U)$ the space of $\mathrm{Ad}(K)$-invariant test functions supported in U. Then there exists a holomorphic map*

$$Q \to \mathcal{F}(U), \quad \lambda \mapsto h_\lambda$$

such that $\Pi_\lambda(h_\lambda)\xi = \xi$.

Proof. Let $V_\xi \subset \mathbb{C}[M\backslash K]$ be the K-module generated by ξ. Let $n := \dim V_\xi$. Let U_0 be a $\mathrm{Ad}(K)$-invariant neighborhood of $\mathbf{1} \in G$ such that $U_0^n \subset U$.

Note that any $h \in \mathcal{F}(U_0)$ induces operators

$$T(\lambda) := \Pi_\lambda(h)|_{V_\xi} \in \mathrm{End}(V_\xi).$$

The compactness of Q allows us to employ uniform Dirac-approximation: we can choose h such that

$$Q \to \mathrm{Gl}(V_\xi), \quad \lambda \mapsto T(\lambda)$$

is defined and holomorphic. Let $n := \dim V_\xi$. By Cayley-Hamilton $T(\lambda)$ is a zero of its characteristic polynomial and hence

$$\mathrm{id}_{V_\xi} = \frac{1}{\det T(\lambda)} \sum_{j=1}^{n} c_j(\lambda) T(\lambda)^j$$

with $c_j(\lambda)$ holomorphic. Set now

$$h_\lambda := \frac{1}{\det T(\lambda)} \sum_{j=1}^{n} c_j(\lambda) \underbrace{h_\lambda * \ldots * h_\lambda}_{\text{j-times}} .$$

Then $Q \ni \lambda \mapsto h_\lambda \in \mathcal{F}(U)$ is holomorphic and $\Pi_\lambda(h_\lambda)\xi = \xi$. $\qquad \square$

For a compactly supported measure ν on G and $f \in \mathcal{S}(G)$ we define $\nu * f \in \mathcal{S}(G)$ by

$$\nu * f(g) = \int_G f(x^{-1}g) \, d\nu(x) .$$

For an element $g \in G$ we denote by δ_g the Dirac delta-distribution at g. Further we view δ_τ as a compactly supported measure on G via the correspondence $\delta_\tau \leftrightarrow \delta_\tau(k) \, dk$.

For each $\tau \in \hat{K}_M$ we define $h_{\lambda,\tau} \in \mathcal{S}(G)$ by

$$(7.4) \qquad\qquad h_{\lambda,\tau} := \delta_\tau * \delta_{a_{t(\tau)}} * h_\lambda .$$

Call a sequence $(c_\tau)_{\tau \in \hat{K}_M}$ rapidly decreasing if

$$\sup_\tau |c_\tau|(1 + |\tau|)^R < \infty$$

for all $R > 0$.

Lemma 7.7. *Let $(c_\tau)_\tau$ be a rapidly decreasing sequence $(c_\tau)_\tau$ and $h_{\lambda,\tau}$ defined as in (7.4). Then*

$$H_\lambda := \sum_{\tau \in \hat{K}_M} c_\tau \cdot h_{\lambda,\tau}$$

is in $\mathcal{S}(G)$ and the assignment $Q \ni \lambda \to H_\lambda \in \mathcal{S}(G)$ is holomorphic.

Proof. Fix $\lambda \in Q$. For simplicity set $H = H_\lambda$, $h_{\lambda,\tau} = h_\tau$.

It is clear that the convergence of H is uniform on compacta and hence $H \in C(G)$. For $u \in \mathcal{U}(\mathfrak{g})$ we record

$$R_u(h_\tau) = \delta_\tau * \delta_{a_{t(\tau)}} * R_u(h)$$

and as a result $H \in C^\infty(G)$. So we do not have to worry about right derivatives. To show that $H \in \mathcal{S}(G)$ we employ Remark 2.10: it remains to show that $H \in \mathcal{R}(G)$, i.e.

$$\tag{7.5} \sup_{g \in \mathfrak{g}} \|g\|^r \cdot |H(g)| < \infty$$

for all $r > 0$. Fix $r > 0$. Write $g = k_1 a k_2$ for some $a \in A$, $k_1, k_2 \in K$. Then

$$\|g\|^r |h_\tau(g)| \leq \|a\|^r \cdot \sup_{k,k' \in K} |h(a_t^{-1} k a k')| \, .$$

Let $Q \subset A$ be a compact set with $\log Q$ convex and $\mathcal{W}$-invariant and such that $\operatorname{supp} h \subset KQK$. We have to determine those $a \in A$ with

$$\tag{7.6} a_t^{-1} K a \cap KQK \neq \emptyset \, .$$

Define $Q_t \subset A$ through $\log Q_t$ being the convex hull of $\mathcal{W}(\log a_t + \log Q)$. Then (7.6) implies that

$$a \in Q_t \, .$$

But this means that $\|a\| << |\tau|^c$ for some $c > 0$, independent of τ. Hence (7.5) is verified and H is indeed in $\mathcal{S}(G)$.

Finally the fact that the assignment $\lambda \mapsto H_\lambda$ is holomorphic follows from the previous Lemma. $\qquad\square$

Theorem 7.8. *Let $Q \subset \mathfrak{a}_\mathbb{C}^*$ be a compact subset. Then there exist a continuous map*

$$Q \times C^\infty(M \backslash K) \to \mathcal{S}(G), \quad (\lambda, v) \mapsto f(\lambda, v)$$

which is holomorphic in the first variable, linear in the second and such that

$$\Pi_\lambda(f(\lambda, v))\xi = v \, .$$

In particular, $\Pi_\lambda(\mathcal{S}(G))V_\lambda = \mathcal{H}_\lambda^\infty$ for all $\lambda \in \mathfrak{a}_\mathbb{C}^*$.

Proof. Let $v \in \mathcal{H}_\lambda^\infty$. Then $v = \sum_\tau c_\tau v_\tau$ with v_τ normalized and $(c_\tau)_\tau$ rapidly decreasing. As $\mathcal{S}(G)$ is stable under left convolution with $C^{-\infty}(K)$ we readily reduce to the case where $v_\tau = \frac{1}{\sqrt{d(\tau)}}\delta_\tau$.

In order to explain the idea of the proof let us first treat the case where the Harish-Chandra module is a multiplicity free K-module. This is for instance satisfied when $G = \mathrm{Sl}(2, \mathbb{R})$.

Recall the numbers $b(\lambda, \tau)$ from Theorem 7.3 and define

$$H_\lambda := \sum_\tau \frac{c_\tau}{\sqrt{d(\tau)} \cdot b(\lambda, \tau)} h_{\lambda, \tau} \, .$$

It follows from Theorem 7.3 and the Lemma above that $Q \ni \lambda \to H_\lambda \in \mathcal{S}(G)$ is defined and holomorphic. By multiplicity one we get that

$$\Pi_\lambda(H_\lambda)\xi = \sum_\tau c_\tau v_\tau \, .$$

and the assertion follows for the multiplicity free case.

Let us move to the general case. For that we employ the more general approximation in Theorem 7.3 and set

$$H'_\lambda = \sum_{\tau \in \hat{K}_M} \frac{c_\tau}{\sqrt{d(\tau)} \cdot b(\lambda, \tau)} h_{\lambda, \tau} \, .$$

Then

$$\Pi_\lambda(H'_\lambda)\xi = \sum_{\tau \in \hat{K}_M} c_\tau v_\tau + R$$

where, given $k > 0$, we can assume that $\|R_\tau\| \leq |c_\tau| \cdot (|\tau| + 1)^{-k}$ for all τ (choose N in Theorem 7.3 big enough). Finally we remove the remainder R_τ by left convolution with $C^{-\infty}(K)$. $\qquad\square$

8. Reduction steps I: extensions, tensoring and induction

In this section we will show that "good" is preserved by induction, tensoring with finite dimensional representations and as well by extensions. We wish to emphasize that these results are not new can be found for instance in [12], Sect. 11.7.

8.1. Extensions

Lemma 8.1. *Let*

$$0 \to U \to L \to V \to 0$$

be an exact sequence of Harish-Chandra modules. If U and V are good, then L is good.

Proof. Let $(\pi, \overline{L})$ be a smooth Fréchet globalization of L. Define a smooth Fréchet globalization $(\pi_U, \overline{U})$ of U by taking the closure of U in $\overline{L}$. Likewise we define a smooth Fréchet globalization $(\pi_V, \overline{V})$ of $V = L/U$ by $\overline{V} := \overline{L}/\overline{U}$. By assumption we have $\overline{U} = \Pi_U(\mathcal{S}(G))U$ and $\overline{V} = \Pi_V(\mathcal{S}(G))V$. As $0 \to \overline{U} \to \overline{L} \to \overline{V} \to 0$ is exact, we deduce that $\Pi(\mathcal{S}(G))L = \overline{L}$, i.e. L is good. $\qquad\square$

As Harish-Chandra modules admit finite composition series we conclude:

Corollary 8.2. *In order to show that all Harish-Chandra modules are good it is sufficient to establish that all irreducible Harish-Chandra modules are good.*

8.2. **Tensoring with finite dimensional representations**

This subsection is devoted to tensoring a Harish-Chandra module with a finite dimensional representation.

Let V be a Harish-Chandra module and V^∞ its minimal globalization. Let (σ, W) be a finite dimensional representation of G. Set

$$\mathbf{V} := V \otimes W$$

and note that $\mathbf{V}$ is a Harish-Chandra module as well. It is our goal to show that the minimal globalization of $\mathbf{V}$ is given by $V^\infty \otimes W$.

Let us fix a covariant inner product $\langle \cdot, \cdot \rangle$ on W. Let $w_1, \ldots, w_k$ be a corresponding orthonormal basis of W. With that we define the $C^\infty(G)$-valued $k \times k$-matrix

$$\mathfrak{S} := (\langle \sigma(g)w_i, w_j \rangle)_{1 \leq i,j \leq k}$$

and record the following:

Lemma 8.3. *With the notation introduced above, the following assertions hold:*

(i) *The map*

$$\mathcal{S}(G)^k \to \mathcal{S}(G)^k, \ \mathbf{f} = (f_1, \ldots, f_k) \mapsto \mathfrak{S}(\mathbf{f})$$

is a linear isomorphism.

(ii) *The map*

$$[C_c^\infty(G)]^k \to [C_c^\infty(G)]^k, \ \mathbf{f} = (f_1, \ldots, f_k) \mapsto \mathfrak{S}(\mathbf{f})\,.$$

is a linear isomorphism.

Proof. First, we observe that the determinant of $\mathfrak{S}$ is 1 and hence $\mathfrak{S}$ is invertible. Second, all coefficients of $\mathfrak{S}$ and $\mathfrak{S}^{-1}$ are of moderate growth, i.e. dominated by a power of $\|g\|$. Both assertions follow. $\square$

Lemma 8.4. *Let V be a Harish-Chandra module and (σ, W) be a finite dimensional representation of G. Then*

$$\mathbf{V}^\infty = V^\infty \otimes W\,.$$

Proof. We denote by $\widetilde{\pi} = \pi \otimes \sigma$ the tensor representation of G on $V^\infty \otimes W$. It is sufficient to show that $v \otimes w_j$ lies in $\widetilde{\Pi}(\mathcal{S}(G))\mathbf{V}$ for all $v \in V^\infty$ and $1 \leq j \leq k$.

Fix $v \in V^\infty$. It is no loss of generality to assume that $j = 1$. By assumption we find $\xi \in V$ and $f \in \mathcal{S}(G)$ such that $\Pi(f)\xi = v$.

We use the previous lemma and obtain an $\mathbf{f} = (f_1, \ldots, f_k) \in \mathcal{S}(G)^k$ such that

$$\mathfrak{S}^t(\mathbf{f}) = (f, 0, \ldots, 0)\,.$$

We claim that

$$\sum_{j=1}^k \widetilde{\Pi}(f_j)(\xi \otimes w_j) = v \otimes w_1\,.$$

In fact, contracting the left hand side with $w_i^* = \langle \cdot, w_i \rangle$ we get that

$$(\mathrm{id} \otimes w_i^*)\left(\sum_{j=1}^k \widetilde{\Pi}(f_j)(\xi \otimes w_j)\right) = \sum_{j=1}^k \int_G f_j(g)\langle \sigma(g)w_j, w_i \rangle \pi(g)\xi \; dg$$

$$= \delta_{1i} \cdot \int_G f(g)\pi(g)\xi \; dg = \delta_{1i} \cdot v$$

and the proof is complete. $\square$

Proposition 8.5. *Let V be a good Harish-Chandra module and (σ, W) be a finite dimensional representation of G. Then $\mathbf{V} = V \otimes W$ is good.*

Proof. It is easy to see that the maximal and minimal globalization of $\mathbf{V}$ coincide. $\square$

Let us call a minimal principal series *pure* if the finite dimensional $P_{\min}$-representation of which it is induced off is trivial on N. We recall that every irreducible Harish-Chandra module can be embedded into a pure minimal principal series and that every pure minimal principal series can be embedded into a module of type $\mathbf{V}_\lambda := V_\lambda \otimes W$ where V_λ

is a spherical principal series and W is a finite dimensional representation of G. As V_λ is good (cf. Corollary 7.5) we thus conclude from Proposition 8.5 and Lemma 4.4 that:

Corollary 8.6. *Every irreducible Harish-Chandra module admits an embedding into a good module.*

8.3. **Induction**

Let $P \supset P_{\min}$ be a parabolic subgroup with Langlands decomposition

$$P = N_P A_P M_P \,.$$

Note that $A_P < A$, $M_P A_P = Z_G(A_P)$ and $N = N_P \rtimes (M_P \cap N)$. For computational purposes it is useful to recall that parabolics P above $P_{\min}$ are parameterized by subsets F of the simple roots Π in $\Sigma(\mathfrak{a}, \mathfrak{n})$. We then often write P_F instead of P, A_F instead of A_P etc. The correspondence $F \leftrightarrow P_F$ is such that

$$A_F = \{a \in A \mid (\forall \alpha \in F)\ a^\alpha = 1\} \,.$$

We make an emphasis on the two extreme cases for F, namely: $P_\emptyset = P_{\min}$ and $P_\Pi = G$.

In the sequel we write $\mathfrak{a}_P$, $\mathfrak{n}_P$ for the Lie algebras of A_P and N_P and denote by $\rho_P \in \mathfrak{a}_P^*$ the usual half sum. Note that $K_P := K \cap M_P$ is a maximal compact subgroup of M_P. Let V_σ be a Harish-Chandra module for M_P and $(\sigma, V_\sigma^\infty)$ its minimal SF-globalization.

For $\lambda \in (\mathfrak{a}_P)_{\mathbb{C}}^*$ we define as before the smooth principal series with parameter (σ, λ) by

$$E_{\sigma,\lambda} = \{f \in C^\infty(G, V_\sigma^\infty)) \,|(\forall\ nam \in P \ \forall\ g \in G)$$

$$f(namg) = a^{\rho_P + \lambda} \sigma(m) f(g)\} \,.$$

and representation $\pi_{\sigma,\lambda}$ by right translations in the arguments of functions in $E_{\sigma,\lambda}$.

In this context we record:

Proposition 8.7. *Let $P \supseteq P_{\min}$ be a parabolic subgroup with Langlands decomposition $P = N_P A_P M_P$. Let V_σ be an irreducible good Harish-Chandra module for M_P. Then for all $\lambda \in (\mathfrak{a}_P)_{\mathbb{C}}^*$ the induced Harish-Chandra module $V_{\sigma,\lambda}$ is good. In particular, $V_{\sigma,\lambda}^\infty = E_{\sigma,\lambda}$.*

Proof. As $\widetilde{V}_{\sigma,\lambda} \simeq V_{\sigma^*,-\lambda}$ it is sufficient to show that $V_{\sigma,\lambda}$ is good.

In the first step we will show that $E_{\sigma,\lambda}$ is the maximal globalization of $V_{\sigma,\lambda}$. To begin with let $N^P := M_P \cap N$ and $A^P := M_P \cap A$. Then $Q := N^P A^P M$ is a minimal parabolic subgroup of M_P. As V_σ is

irreducible, we find an embedding of V_σ into a pure minimal principal series module of M_P, say I_σ:

$$I_\sigma = \mathrm{Ind}_Q^{M_P}(\mathbf{1} \otimes (\mu + \rho^P) \otimes \gamma)$$

with $\mu \in (\mathfrak{a}^P)^*_{\mathbb{C}}$ and (γ, U_γ) an irreducible representation of M. Then $L^2(U_\gamma \times_M K_P)$ is a Hilbert globalization of I_σ and we denote by $\mathcal{H}_\sigma$ the closure of V_σ in $L^2(U_\gamma \times_M K_P)$. As V_σ is good, it follows that $V_\sigma^\infty = \mathcal{H}_\sigma^\infty$. With $\mathcal{H}_\sigma$ we obtain a Hilbert model for $V_{\sigma,\lambda}$ namely $\mathcal{H}_{\sigma,\lambda} = L^2(\mathcal{H}_\sigma \times_{K_P} K)$. Notice that the smooth vectors of $\mathcal{H}_{\sigma,\lambda}$ coincide with $E_{\sigma,\lambda}$.

We proceed with double induction and obtain an embedding of $V_{\sigma,\lambda}$ into a tensor product module $\mathbf{V} = V_{\lambda+\mu} \otimes W$, where $V_{\lambda+\mu}$ is a spherical minimal principal series with parameter $\lambda + \mu \in \mathfrak{a}_{\mathbb{C}}*$ and W is a finite dimensional irreducible representation whose A-highest weight space is isomorphic to U_γ as an M-module. We view $\mathbf{V}$ as a minimal principal series and endow with the Hilbert structure induced from the compact model $\mathbf{H} = L^2(W \otimes_M K)$. Observe that the embedding $\mathcal{H}_{\sigma,\lambda}$ to $\mathbf{H}$ is isometric. As $\mathbf{V}$ is good we get that the maximal globalization of $V_{\sigma,\lambda}$ is the closure of $V_{\sigma,\lambda}$ in $\mathbf{V}^\infty$. From our discussion it follows that this closure is $\mathcal{H}_{\sigma,\lambda}^\infty = E_{\sigma,\lambda}$.

To conclude the proof we need to show that $E_{\sigma,\lambda}$ coincides with the minimal globalization of $V_{\sigma,\lambda}$ as well. We proceed dually: start from the realization of V_σ as a quotient of a minimal principal series J_σ of M_P etc. As before we will end up with a Hilbert model $\hat{H}_{\sigma,\lambda}$ for $V_{\sigma,\lambda}$ with $\hat{H}_{\sigma,\lambda}^\infty = E_{\sigma,\lambda}$ and an orthogonal projection of some Hilbert globalization $\hat{\mathbf{H}}$ of some good tensor product module onto $\hat{H}_{\sigma,\lambda}$. Hence Lemma 4.2 implies that $E_{\sigma,\lambda}$ equals the minimal globalization. $\qquad\square$

9. **Reduction steps II: deformation theory**

The goal of this section is to show that all Harish-Chandra modules are good. We already know that every irreducible Harish-Chandra modules V can be written as a quotient U/H where U is good. Suppose that H is in fact a kernel of an intertwiner $I : U \to W$ with W good. Suppose in addition that we can deform $I : U \to W$ holomorphically (as to be made precise in the following section). Then, provided U and W are good we will show that $\mathrm{im}\, I \simeq U/H$ is good. In view of the Langlands-classification the assertion that every Harish-Chandra module is good then reduces to the case of discrete series representations which we established in Proposition 6.1.

9.1. **Deformations**

For a complex manifold D and a Harish-Chandra module U we write $\mathcal{O}(D, U)$ for the space of maps $f : D \to U$ such that for all $\xi \in \widetilde{U}$ the contraction $\xi \circ f$ is holomorphic. Henceforth we will use D exclusively for the open unit disc.

By a holomorphic family of Harish-Chandra modules (parameterized by D) we understand a family of Harish-Chandra modules $(U_s)_{s \in D}$ such that:

(i) For all $s \in D$ one has $U_s = U_0 =: U$ as K-modules.
(ii) For all $X \in \mathfrak{g}$, $v \in U$ and $\xi \in \widetilde{U}$ the map $s \mapsto \xi(X_s \cdot v)$ is holomorphic. Here we use X_s for the action of X in U_s.

Given a holomorphic family $(U_s)_{s \in D}$ we form $\mathcal{U} := \mathcal{O}(D, U)$ and endow it with the following $(\mathfrak{g}, K)$-structure: for $X \in \mathfrak{g}$ and $f \in \mathcal{U}$ we set

$$(X \cdot f)(s) := X_s \cdot f(s) .$$

Of particular interest are the Harish-Chandra modules $\mathbf{U}_k := \mathcal{U}/s^k \mathcal{U}$ for $k \in \mathbb{N}$. To get a feeling for this objects let us discuss a few examples for small k.

Example 9.1. *(a) For $k = 1$ the constant term map*

$$\mathbf{U}_1 \to U, \quad f + s\mathcal{U} \mapsto f(0)$$

is an isomorphism of $(\mathfrak{g}, K)$-modules.
(b) For $k = 2$ we observe that the map

$$\mathbf{U}_2 \to U \oplus U, \quad f + s^2 \mathcal{U} \mapsto (f(0), f'(0))$$

provides an isomorphism of K-modules. The resulting $\mathfrak{g}$-action on the right hand side is twisted and given by

$$X \cdot (u_1, u_2) = (X u_1, X u_2 + X' u_1)$$

where

$$X'u := \frac{d}{ds}\Big|_{s=0} X_s \cdot u .$$

Let us remark that $X' = 0$ for all $X \in \mathfrak{k}$.
We notice that $\mathbf{U}_2$ features the submodule $s\mathcal{U}/s^2\mathcal{U}$ which corresponds to $\{0\} \oplus U$ in the above trivialization. The corresponding quotient $(\mathcal{U}/s^2\mathcal{U})/(s\mathcal{U}/s^2\mathcal{U})$ identifies with $U \oplus U/\{0\} \oplus U \simeq U$. In particular $\mathcal{U}/s^2\mathcal{U}$ is good if U is good by the extension Lemma 8.1.

From the previous discussion it follows that $\mathbf{U}_k$ is good for all $k \in \mathbb{N}_0$ provided that U is good.

Let now W be another Harish-Chandra module and $\mathcal{W}$ a holomorphic deformation of W as above. By a morphism $\mathcal{I} : \mathcal{U} \to \mathcal{W}$ we understand a family of $(\mathfrak{g}, K)$-maps $I_s : U_s \to W_s$ such that for all $u \in U$ and $\xi \in W^*$ the assignments $s \mapsto \xi(I_s(u))$ are holomorphic. Let us write I for I_0 set $I' := \frac{d}{ds}\big|_{s=0} I_s$ etc. We set $H := \ker I$.

We now make two additional assumptions on our holomorphic family of intertwiners:

- I_s is invertible for all $s \neq 0$.
- There exists a $k \in \mathbb{N}_0$ such that $J(s) := s^k I_s^{-1}$ is holomorphic on D.

If these conditions are satisfied, then we call $I : U \to W$ *holomorphically deformable.*

For all $m \in \mathbb{N}$ we write $\mathbf{I}_m : \mathbf{U}_m \to \mathbf{W}_m$ for the intertwiner induced by $\mathcal{I}$. Likewise we define $\mathbf{J}_m$.

Example 9.2. *In order to get a feeling for the intertwiners $\mathbf{I}_m$ let us consider the example $\mathbf{I}_2 : \mathbf{U}_2 \to \mathbf{W}_2$. In trivializing coordinates this map is given by*

$$\mathbf{I}_2(u_1, u_2) = (I(u_1), I(u_2) + I'(u_1)).$$

We set $\mathbf{H}_m := \ker \mathbf{I}_m \subset \mathbf{U}_m$. For $m < n$ we view $\mathbf{U}_m$ as a K-submodule of $\mathbf{U}_n$ via the inclusion map

$$\mathbf{U}_m \to \mathbf{U}_n, \quad f + s^m \mathcal{U} \mapsto \sum_{j=0}^{m-1} \frac{f^{(j)}(0)}{j!} s^j + s^n \mathcal{U}.$$

We write $p_{n,m} : \mathbf{U}_n \to \mathbf{U}_m$ for the reverting projection (which are $(\mathfrak{g}, K)$-morphisms).

The following Lemma is related to an observation of Casselman as recorded in [12], 11.7.9.

Lemma 9.3. *The morphism*

$$\mathbf{I}_{2k}|_{\mathbf{H}_k + s^k \mathcal{U}/s^{2k}\mathcal{U}} : \mathbf{H}_k + s^k \mathcal{U}/s^{2k}\mathcal{U} \to s^k \mathcal{W}/s^{2k}\mathcal{W}$$

is onto. Moreover, its kernel is given by $s^k \mathbf{H}_k \subset s^k \mathcal{U}/s^{2k}\mathcal{U}$.

Proof. Clearly, $\mathbf{I}_{2k}^{-1}(s^k\mathcal{W}/s^{2k}\mathcal{W}) = \mathbf{H}_k + s^k\mathcal{U}/s^{2k}\mathcal{U}$ and hence the map is defined. Let us check that it is onto. Let $[w] \in s^k\mathcal{W}/s^{2k}\mathcal{W}$ and $w \in s^k\mathcal{W}$ be a representative. Note that $s^{-k}\mathcal{J}|_{s^k\mathcal{W}} : s^k\mathcal{W} \to \mathcal{U}$ is defined. Set $u := s^{-k}\mathcal{J}(w)$ and write $[u]$ for its equivalence class in $\mathbf{U}_{2k}$. Then $\mathbf{I}_{2k}([u]) = [w]$ and the map is onto.

A simple verification shows that $s^k \mathbf{H}_k$ lies in the kernel. Hence by considering the surjective map K-type by K-type we arrive that it equals the kernel by dimension count. $\square$

If we set $V_3 := \mathbf{H}_k + s^k\mathcal{U}/s^{2k}\mathcal{U}$, $V_2 := s^k\mathcal{U}/s^{2k}\mathcal{U}$ and $V_1 := s^k\mathbf{H}_k$, the previous Lemma implies an inclusion chain

$$V_1 \subset V_2 \subset V_3$$

with

$$V_2/V_1 \simeq \mathbf{U}_k/\mathbf{H}_k, \qquad V_2 \simeq \mathbf{U}_k \qquad \text{and} \qquad V_3/V_1 \simeq \mathbf{W}_k\,.$$

Hence in combination with the squeezing Lemma 4.5 we obtain that $\mathbf{U}_k/\mathbf{H}_k$ is good if U and W are good.

We wish to show that U/H is good. This follows now by iteration and it is enough to consider the case $k = 2$ in more detail. Write $H_{2,1} := p_{2,1}(\mathbf{H}_2)$ for the projection of $\mathbf{H}_2$ to $\mathbf{U}_1 \simeq U$. Note that $H_{2,1}$ is a submodule of H. We arrive at the exact sequence

$$0 \to U/H \simeq sU/sH \to \mathbf{U}_2/\mathbf{H}_2 \to U/H_{2,1} \to 0\,.$$

But U/H is a quotient of $U/H_{2,1}$. Thus putting an SF-topology on U we get one on H, $\mathbf{U}_2$, $\mathbf{H}_2$, $\mathbf{U}_2/\mathbf{H}_2$ and $U/H_{2,1}$. As a result the induced topology on U/H is both a sub and a quotient of the good topology on $\mathbf{U}_2/\mathbf{H}_2$. Hence U/H is good.

We summarize our discussion.

Proposition 9.4. *Suppose that $I : U \to W$ is an intertwiner of good Harish-Chandra modules which allows holomorphic deformations $\mathcal{I} : \mathcal{U} \to \mathcal{W}$. Then $\operatorname{im} I$ is good.*

9.2. **All Harish-Chandra modules are good**

In this subsection we will prove that all Harish-Chandra modules are good. In view of the deformation result (Prop. 9.4) and the Langlands classification we are readily reduced to the case of discrete series representation.

Theorem 9.5. *All Harish-Chandra modules are good.*

Proof. Let V be a Harish-Chandra module. We have to show that V is good. In view of Corollary 8.2, we may assume that V is irreducible. Next we use Langland's classification (see [8],Ch. VIII, Th. 8.54) and combine it with our Propositions on deformation 9.4 and induction 8.7. This reduces to the case where V is tempered. However, the case of tempered readily reduces to square integrable ([11], Ch. 5, Prop. 5.2.5). The case of square integrable Harish-Chandra modules was established in Proposition 6.1. $\qquad\qquad\square$

10. **Applications**

10.1. **Lifting $(\mathfrak{g}, K)$-morphisms**

Let (π, E) be a representation of G on a complete topological vector space E. Let us call (π, E) an $\mathcal{S}(G)$-*representation* if the natural action of $C_c^\infty(G)$ on E extends to a separately continuous action of $\mathcal{S}(G)$ on E. Some typical examples we have in mind are smooth functions of moderate growth on certain homogeneous spaces. Let us mention a few.

Example 10.1. *(a) Let $\Gamma < G$ be a lattice, that is a discrete subgroup with cofinite volume. Reduction theory (Siegel sets) allows us to control "infinity" of the quotient $Y := \Gamma \backslash G$ and leads to a natural notion of moderate growth. For every $\alpha > 0$ there is a natural SF-module $C_\alpha^\infty(Y)$ of smooth functions on Y with growth rate at most α. The smooth functions of moderate growth $C_{\mathrm{mod}}^\infty(Y) = \lim_{\alpha \to \infty} C_\alpha^\infty(Y)$ become a complete inductive limit of the SF-spaces $C_\alpha^\infty(Y)$. Hence $\mathcal{S}(G)$ acts on $C_{\mathrm{mod}}^\infty(Y)$.*

The space of K and $\mathcal{Z}(\mathfrak{g})$-finite elements in $C_{\mathrm{mod}}^\infty(Y)$ is referred to as the space of automorphic forms on Y.

(b) Let $H < G$ be a symmetric subgroup, i.e. an open subgroup of the fixed point set of an involutive automorphism of G. We refer to $X := H \backslash G$ as a semisimple symmetric space. The Cartan-decomposition of X allows us to control growth on X and yields natural SF-modules $C_\alpha^\infty(X)$ of smooth functions with growth rate at most α. As before one obtains $C_{\mathrm{mod}}^\infty(X) = \lim_{\alpha \to \infty} C_\alpha^\infty(X)$ a natural complete $\mathcal{S}(G)$-module of functions with moderate growth.

If (π_1, E_1), (π_2, E_2) are two represesentations, then we denote by $\mathrm{Hom}_G(E_1, E_2)$ for the space of continuous G-equivariant linear maps from E_1 to E_2.

Proposition 10.2. *Let V be a Harish-Chandra module and V^∞ its unique SF-globalization. Then for an $\mathcal{S}(G)$-representation (π, E) of G the linear map*

$$\mathrm{Hom}_G(V^\infty, E) \to \mathrm{Hom}_{(\mathfrak{g}, K)}(V, E_K), \quad T^\infty \mapsto T := T^\infty|_V$$

is a linear isomorphism.

Proof. It is clear that the map is injective and move on to onto-ness. Let us write λ, resp. Λ, for the representation of G, resp. $\mathcal{S}(G)$, on V^∞. Let $v \in V^\infty$. Then we find $f \in \mathcal{S}(G)$ such that $v = \Lambda(f)w$ for some $w \in V$. We claim that

$$T^\infty(v) := \Pi(f)T(w)$$

defines a linear operator. In order to show that this definition makes sense we have to show that $T^\infty(v) = 0$ provided that $\Lambda(f)w = 0$. Let $\xi \in (E^*)_K$ and $\mu := \xi \circ T \in \widetilde{V}$. We consider two distributions on G, namely

$$\Theta_1(\phi) := \xi(\Pi(\phi)T(w)) \quad \text{and} \quad \Theta_2(\phi) := \mu(\Lambda(\phi)w) \qquad (\phi \in C_c^\infty(G)).$$

We claim that $\Theta_1 = \Theta_2$. In fact, both distributions are $\mathcal{Z}(\mathfrak{g})-$ and $K \times K$-finite. Hence they are represented by analytic functions on G and thus uniquely determined by their derivatives on K. The claim follows.

It remains to show that T is continuous. We recall the construction of the minimal SF-globalization of V, namely $V^\infty = \mathcal{S}(G)^k/\mathcal{S}(G)_{\mathbf{v}}$. As the action of $\mathcal{S}(G)$ on E is separately continuous, the continuity of T^∞ follows. $\qquad\square$

10.2. Automatic continuity

For a Harish-Chandra module V we denoted by V^* its algebraic dual. Note that V^* is naturally a module for $\mathfrak{g}$.

If $\mathfrak{h} < \mathfrak{g}$ is a subalgebra, then we write $(V^*)^{\mathfrak{h}}$, resp. $(V^*)^{\mathfrak{h}-\text{fin}}$, for the space of $\mathfrak{h}$-fixed, resp. $\mathfrak{h}$-finite, algebraic linear functionals on V.

We call a sublagebra $\mathfrak{h} < \mathfrak{g}$ a *(strong) automatic continuity* subalgebra *((S)AC-subalgebra* for short) if for all Harish-Chandra modules V one has

$$(V^*)^{\mathfrak{h}} \subset (V^\infty)^* \quad \text{resp.} \quad (V^*)^{\mathfrak{h}-\text{fin}} \subset (V^\infty)^*.$$

Problem 10.3. *(a) Is it true that $\mathfrak{h}$ is AC if and only if $\langle \exp \mathfrak{h} \rangle < G$ has an open orbit on $G/P_{\min}$.*
(b) Is it true that $\mathfrak{h}$ is SAC if $[\mathfrak{h}, \mathfrak{h}]$ is AC?

The following examples of (S)AC-subalgebras are known:

- $\mathfrak{n}$, the Lie algebra of an Iwasawa N-subgroup, is AC and $\mathfrak{a}+\mathfrak{n}$, the Lie algebra of an Iwasasaw AN-subgroup, is SAC. (Casselman).
- Symmetric subalgebras, i.e. fixed point sets of involutive automorphisms of $\mathfrak{g}$, are AC (Brylinski, Delorme, van den Ban; cf. [2], [1]).

Here we only wish to discuss Casselman's result. We recall the definition of the Casselman-Jacquet module $j(V) = \bigcup_{k \in \mathbb{N}_0}(V/\mathfrak{n}^k V)^*$ and note that $j(V) = (V^*)^{\mathfrak{a}+\mathfrak{n}-\text{fin}}$.

Theorem 10.4. (Casselman) *Let $\mathfrak{n}$ be the Lie algebra of an Iwasawa N-subgroup of G and $\mathfrak{a}+\mathfrak{n}$ the Lie algebra of an Iwasawa AN-subgroup. Then $\mathfrak{n}$ is an AC and $\mathfrak{a}+\mathfrak{n}$ is SAC. In particular, for all Harish-Chandra modules V one has $j(V) \subset (V^\infty)^*$.*

Proof. Let us first prove that $\mathfrak{a}+\mathfrak{n}$ is SAC. Let V be a Harish-Chandra module and $0 \neq \lambda \in j(V)$. By definition there exists a $k \in \mathbb{N}$ such that $\lambda \in (V/\mathfrak{n}^k V)^*$. Write (σ, U_σ) for the finite dimensional representation of $P_{\min}$ on $V/\mathfrak{n}^k V$ and denote by I_σ the corresponding induced Harish-Chandra module. Note that $I_\sigma^\infty = C^\infty(G \times_{P_{\min}} U_\sigma)$.

Applying Frobenius reciprocity to the identity morphism $V/\mathfrak{n}^k V \to U$ yields a non-trivial $(\mathfrak{g}, K)$-morphism $T : V \to I_\sigma$ (cf. [11], 4.2.2). Now T lifts to a continuous G-map $T^\infty : V^\infty \to I_\sigma^\infty$ by Proposition 10.2. If $\mathrm{ev} : I_\sigma^\infty \to U_\sigma$ denotes the evaluation map at the identity, then $\lambda^\infty := \lambda \circ \mathrm{ev} \circ T^\infty$ provides a continuous extension of λ to V^∞.

The proof that $\mathfrak{n}$ is AC goes along the same lines. $\qquad\square$

10.3. **Lifting of holomorphic families of $(\mathfrak{g}, K)$-maps**

We wish to give a version of lifting (cf. Proposition 10.2) which depends holomorphically on parameters.

Theorem 10.5. *Let $P = N_P A_P M_P$ be a parabolic subgroup and V_σ a Harish-Chandra module of an SAF-representation of M_P. Let (π, E) be an SF-representation. Suppose that there is a family of $(\mathfrak{g}, K)$-intertwiners $T_\lambda : V_{\sigma,\lambda} \to E_K$ such that for all $v \in \mathbb{C}[V_\sigma \times_{K_P} K]$ and $\xi \in E^*$ the assignments $\lambda \mapsto \xi(T_\lambda(v))$ are holomorphic. Then for all $v \in C^\infty((V_\sigma^\infty \times_{K_P} K)^\infty)$ and $\xi \in E^*$, the maps*

$$(\mathfrak{a}_P)_{\mathbb{C}}^* \ni \lambda \mapsto \xi(T_\lambda^\infty(v)) \in \mathbb{C}$$

are holomorphic.

The proof is an immediate consequence of the analogue to Theorem 7.3 and Theorem 7.8 for the induced family considered.

Let $P = N_P A_P M_P$ be a parabolic above $P_{\min}$. We fix an SAF-representation $(\sigma, V_\sigma^\infty)$ of M_P and write V_σ for the corresponding Harish-Chandra module.

As K-modules we identify all $V_{\sigma,\lambda}$ with $V := \mathbb{C}[V_\sigma \times_{K_P} K]$. Note that V_σ is a K_P-quotient of some $\mathbb{C}[K_P]^m$, $m \in \mathbb{N}$. Double induction gives an identification of V as a K-quotient of $\mathbb{C}[K]^m$. Note that $C^\infty(K)^m$ induces the unique SF-topology on V^∞. For each τ we χ_τ for its character and $\delta_{\sigma,\tau}$ for the orthogonal projection of $\underbrace{(\chi_\tau, \ldots, \chi_\tau)}_{m-times}$ to $V[\tau]$, the τ-isotypical part of V.

Theorem 10.6. *Let $P = N_P A_P M_P$ be a parabolic subgroup and V_σ an irreducible unitarizable Harish-Chandra module for M_P. Let $Q \subset (\mathfrak{a}_P)^*_\mathbb{C}$ be a compact subset and $N > 0$. Then there exists $\xi \in \mathbb{C}[V_\sigma \times_{K_P} K]$ and constants $c_1, c_2 > 0$ such that for all $\tau \in \hat{K}$, $\lambda \in Q$, there exists $a_\tau \in A$, independent of λ, with $\|a_\tau\| \leq (1+|\tau|)^{c_1}$ and numbers $b_\sigma(\lambda, \tau) \in \mathbb{C}$ such that*

$$\|[\pi_{\sigma,\lambda}(a_\tau)\xi]_\tau - b_\sigma(\lambda, \tau)\delta_{\sigma,\tau}\| \leq \frac{1}{(|\tau| + 1)^{N+c_2}}$$

and

$$|b_\sigma(\lambda, \tau)| \geq \frac{1}{(|\tau| + 1)^{c_2}}\,.$$

Here $\|\cdot\|$ refers to the continuous norm on V induced by the realization of V as a quotient of $C[K]^m \subset L^2(K)^m$.

Proof. Let us first discuss the case where $P = P_{\min}$ and σ is finite dimensional. Here the assertion is a simple modification of Theorem 7.3.

As for the general case we identify V_σ as a quotient of a minimal principal series for M_P. Using double induction we can write the $V_{\sigma,\lambda}$'s consistently as quotients of such minimal principal series. The assertion follows. $\qquad\square$

As a consequence we get an extension of Theorem 7.8.

Theorem 10.7. *Let $Q \subset (\mathfrak{a}_P)^*_\mathbb{C}$ be a compact subset. Then there exist a continuous map*

$$Q \times C^\infty(V_\sigma^\infty \times_{K_P} K) \to \mathcal{S}(G), \quad (\lambda, v) \mapsto f(\lambda, v)$$

which is holomorphic in the first variable, linear in the second and such that

$$\Pi_{\sigma,\lambda}(f(\lambda, v))\xi = v\,.$$

References

[1] E. van den Ban and P. Delorme, *Quelques propriétés des représentations sphériques pour les espaces symétriques réductifs*, J. Funct. Anal. **80** (1988), no. **2**, 284–307.

[2] J.-L. Brylinski and P. Delorme, *Vecteurs distributions H-invariants pour les séries principales généralisées d'espaces symétriques réductifs et prolongement méromorphe d'intégrales d'Eisenstein*, Invent. Math. **109** (1992), no. **3**, 619–664.

[3] W. Casselman, *Canonical extensions of Harish-Chandra modules to representations of G*, Canad. J. Math. **41** (1989), no. **3**, 385–438.

[4] J. Cheeger, M. Gromov and M. Taylor, *Finite Propagation Speed, Kernel Estimates for Functions of the Laplace Operator, and the Geometry of Complete Riemannian Manifolds*, J. Differential Geom. **17** (1982), 15–53.

[5] J. Dixmier and P. Malliavin, *Factorisations de fonctions et de vecteurs indéfiniment différentiables*, Bull. Sci. Math. **(2) 102** (1978), no. **4**, 307–330.

[6] L. Garding, *Vectors analytiques dans le représentations des groupes de Lie*, Bulletin de la S.M.F, tome **88** (1960), 73–93

[7] H. Gimperlein, B. Krötz and C. Lienau, *Analytic factorization of Lie group representations*, arXiv:0910.0177, submitted

[8] A.W. Knapp, *Representation theory of semisimple groups. An overview based on examples*, Reprint of the 1986 original. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 2001.

[9] W. Soergel, *An irreducible not admissible Banach representation of* $SL(2, R)$, Proc. Amer. Math. Soc. **104** (1988), no. **4**, 1322–1324.

[10] E.C. Titchmarsh, *The theory of functions*, Oxford Univ. Press, 1986

[11] N. Wallach, *Real reductive groups I*, Academic Press 1988

[12] N. Wallach, *Real reductive groups II*, Academic Press 1992

[13] G. Warner, *Harmonic analysis on semi-simple Lie groups. I*, Die Grundlehren der mathematischen Wissenschaften, Band **188**, Springer-Verlag, New York-Heidelberg, 1972

School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel,, email:bernstei@math.tau.ac.il

Leibniz Universität Hannover,, Institut für Analysis, Welfengarten 1, D-30167 Hannover, email: kroetz@math.uni-hannover.de